

# Free $A_\infty$ -categories

Volodymyr Lyubashenko\* and Oleksandr Manzyuk†

February 17, 2008

## Abstract

For a differential graded  $\mathbb{k}$ -quiver  $\mathcal{Q}$  we define the free  $A_\infty$ -category  $\mathcal{FQ}$  generated by  $\mathcal{Q}$ . The main result is that the restriction  $A_\infty$ -functor  $A_\infty(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A})$  is an equivalence, where objects of the last  $A_\infty$ -category are morphisms of differential graded  $\mathbb{k}$ -quivers  $\mathcal{Q} \rightarrow \mathcal{A}$ .

$A_\infty$ -categories defined by Fukaya [Fuk93] and Kontsevich [Kon95] are generalizations of differential graded categories for which the binary composition is associative only up to a homotopy. They also generalize  $A_\infty$ -algebras introduced by Stasheff [Sta63, II].  $A_\infty$ -functors are the corresponding generalizations of usual functors, see e.g. [Fuk93, Kel01]. Homomorphisms of  $A_\infty$ -algebras (e.g. [Kad82]) are particular cases of  $A_\infty$ -functors.  $A_\infty$ -transformations are certain coderivations. Examples of such structures are encountered in studies of mirror symmetry (e.g. [Kon95, Fuk02]) and in homological algebra.

For an  $A_\infty$ -category there is a notion of units up to a homotopy (homotopy identity morphisms) [Lyu03]. Given two  $A_\infty$ -categories  $\mathcal{A}$  and  $\mathcal{B}$ , one can construct a third  $A_\infty$ -category  $A_\infty(\mathcal{A}, \mathcal{B})$ , whose objects are  $A_\infty$ -functors  $f : \mathcal{A} \rightarrow \mathcal{B}$ , and morphisms are  $A_\infty$ -transformations between such functors (Fukaya [Fuk02], Kontsevich and Soibelman [KS02, KS], Lefèvre-Hasegawa [LH03], as well as [Lyu03]). This allows to define a 2-category, whose objects are unital  $A_\infty$ -categories, 1-morphisms are unital  $A_\infty$ -functors and 2-morphisms are equivalence classes of natural  $A_\infty$ -transformations [Lyu03]. We continue to study this 2-category.

The notations and conventions are explained in the first section. We also describe  $A_N$ -categories,  $A_N$ -functors and  $A_N$ -transformations – truncated at  $N < \infty$  versions of  $A_\infty$ -categories. For instance,  $A_1$ -categories and  $A_1$ -functors are differential graded  $\mathbb{k}$ -quivers and their morphisms. However,  $A_1$ -transformations bring new 2-categorical features to the theory. In particular, for any differential graded  $\mathbb{k}$ -quiver  $\mathcal{Q}$  and any  $A_\infty$ -category

---

\*Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivska st., Kyiv-4, 01601 MSP, Ukraine; lub@imath.kiev.ua

†Fachbereich Mathematik, Postfach 3049, 67653 Kaiserslautern, Germany; manzyuk@mathematik.uni-kl.de

$\mathcal{A}$  there is an  $A_\infty$ -category  $A_1(\mathcal{Q}, \mathcal{A})$ , whose objects are morphisms of differential graded  $\mathbb{k}$ -quivers  $\mathcal{Q} \rightarrow \mathcal{A}$ , and morphisms are  $A_1$ -transformations. We recall the terminology related to trees in Section 1.7.

In the second section we define the free  $A_\infty$ -category  $\mathcal{FQ}$  generated by a differential graded  $\mathbb{k}$ -quiver  $\mathcal{Q}$ . We classify functors from a free  $A_\infty$ -category  $\mathcal{FQ}$  to an arbitrary  $A_\infty$ -category  $\mathcal{A}$  in Proposition 2.3. In particular, the restriction map gives a bijection between the set of strict  $A_\infty$ -functors  $\mathcal{FQ} \rightarrow \mathcal{A}$  and the set of morphisms of differential graded  $\mathbb{k}$ -quivers  $\mathcal{Q} \rightarrow (\mathcal{A}, m_1)$  (Corollary 2.4). We classify chain maps into complexes of transformations whose source is a free  $A_\infty$ -category in Proposition 2.8. Description of homotopies between such chain maps is given in Corollary 2.10. Assuming in addition that  $\mathcal{A}$  is unital, we obtain our main result: the restriction  $A_\infty$ -functor  $\text{restr} : A_\infty(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A})$  is an equivalence (Theorem 2.12).

In the third section we interpret  $A_\infty(\mathcal{FQ}, \_)$  and  $A_1(\mathcal{Q}, \_)$  as strict  $A_\infty^u$ -2-functors  $A_\infty^u \rightarrow A_\infty^u$ . Moreover, we interpret  $\text{restr} : A_\infty(\mathcal{FQ}, \_) \rightarrow A_1(\mathcal{Q}, \_)$  as an  $A_\infty^u$ -2-equivalence. In this sense the  $A_\infty$ -category  $\mathcal{FQ}$  represents the  $A_\infty^u$ -2-functor  $A_1(\mathcal{Q}, \_)$ . This is the 2-categorical meaning of freeness of  $\mathcal{FQ}$ .

## 1. Conventions and preliminaries

We keep the notations and conventions of [Lyu03, LO02], sometimes without explicit mentioning. Some of the conventions are recalled here.

We assume as in [Lyu03, LO02] that most quivers,  $A_\infty$ -categories, etc. are small with respect to some universe  $\mathcal{U}$ .

The ground ring  $\mathbb{k} \in \mathcal{U}$  is a unital associative commutative ring. A  $\mathbb{k}$ -module means a  $\mathcal{U}$ -small  $\mathbb{k}$ -module.

We use the right operators: the composition of two maps (or morphisms)  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  is denoted  $fg : X \rightarrow Z$ ; a map is written on elements as  $f : x \mapsto xf = (x)f$ . However, these conventions are not used systematically, and  $f(x)$  might be used instead.

$\mathbb{Z}$ -graded  $\mathbb{k}$ -modules are functions  $X : \mathbb{Z} \ni d \mapsto X^d \in \mathbb{k}\text{-mod}$ . A simple computation shows that the product  $X = \prod_{\iota \in I} X_\iota$  in the category of  $\mathbb{Z}$ -graded  $\mathbb{k}$ -modules of a family  $(X_\iota)_{\iota \in I}$  of objects  $X_\iota : d \mapsto X_\iota^d$  is given by  $X : \mathbb{Z} \ni d \mapsto X^d = \prod_{\iota \in I} X_\iota^d$ . Everywhere in this article the product of graded  $\mathbb{k}$ -modules means the above product.

If  $P$  is a  $\mathbb{Z}$ -graded  $\mathbb{k}$ -module, then  $sP = P[1]$  denotes the same  $\mathbb{k}$ -module with the grading  $(sP)^d = P^{d+1}$ . The “identity” map  $P \rightarrow sP$  of degree  $-1$  is also denoted  $s$ . The map  $s$  commutes with the components of the differential in an  $A_\infty$ -category ( $A_\infty$ -algebra) in the following sense:  $s^{\otimes n} b_n = m_n s$ .

Let  $\mathbf{C} = \mathbf{C}(\mathbb{k}\text{-mod})$  denote the differential graded category of complexes of  $\mathbb{k}$ -modules. Actually, it is a symmetric closed monoidal category.

The cone of a chain of a chain map  $\alpha : P \rightarrow Q$  of complexes of  $\mathbb{k}$ -modules is the graded  $\mathbb{k}$ -module  $\text{Cone}(\alpha) = Q \oplus P[1]$  with the differential  $(q, ps)d = (qd^Q + p\alpha, psd^{P[1]}) = (qd^Q + p\alpha, -pd^P s)$ .

**1.1.  $A_N$ -categories.** For a positive integer  $N$  we define some  $A_N$ -notions similarly to the case  $N = \infty$ . We may say that all data, equations and constructions for  $A_N$ -case are the same as in  $A_\infty$ -case (e.g. [Lyu03]), however, taken only up to level  $N$ .

A *differential graded  $\mathbb{k}$ -quiver*  $\mathcal{Q}$  is the following data: a  $\mathcal{U}$ -small set of objects  $\text{Ob } \mathcal{Q}$ ; a chain complex of  $\mathbb{k}$ -modules  $\mathcal{Q}(X, Y)$  for each pair of objects  $X, Y$ . A morphism of differential graded  $\mathbb{k}$ -quivers  $f : \mathcal{Q} \rightarrow \mathcal{A}$  is given by a map  $f : \text{Ob } \mathcal{Q} \rightarrow \text{Ob } \mathcal{A}$ ,  $X \mapsto Xf$  and by a chain map  $\mathcal{Q}(X, Y) \rightarrow \mathcal{A}(Xf, Yf)$  for each pair of objects  $X, Y$  of  $\mathcal{Q}$ . The category of differential graded  $\mathbb{k}$ -quivers is denoted  $A_1$ .

The category of  $\mathcal{U}$ -small graded  $\mathbb{k}$ -linear quivers, whose set of objects is  $S$ , admits a monoidal structure with the tensor product  $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A} \otimes \mathcal{B}$ ,  $(\mathcal{A} \otimes \mathcal{B})(X, Y) = \bigoplus_{Z \in S} \mathcal{A}(X, Z) \otimes_{\mathbb{k}} \mathcal{B}(Z, Y)$ . In particular, we have tensor powers  $T^n \mathcal{A} = \mathcal{A}^{\otimes n}$  of a given graded  $\mathbb{k}$ -quiver  $\mathcal{A}$ , such that  $\text{Ob } T^n \mathcal{A} = \text{Ob } \mathcal{A}$ . Explicitly,

$$T^n \mathcal{A}(X, Y) = \bigoplus_{X=X_0, X_1, \dots, X_n=Y \in \text{Ob } \mathcal{A}} \mathcal{A}(X_0, X_1) \otimes_{\mathbb{k}} \mathcal{A}(X_1, X_2) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \mathcal{A}(X_{n-1}, X_n).$$

In particular,  $T^0 \mathcal{A}(X, Y) = \mathbb{k}$  if  $X = Y$  and vanishes otherwise. The graded  $\mathbb{k}$ -quiver  $T^{\leq N} \mathcal{A} = \bigoplus_{n \geq 0}^N T^n \mathcal{A}$  is called the restricted tensor coalgebra of  $\mathcal{A}$ . It is equipped with the cut comultiplication

$$\begin{aligned} \Delta : T^{\leq N} \mathcal{A}(X, Y) &\rightarrow \bigoplus_{Z \in \text{Ob } \mathcal{A}} T^{\leq N} \mathcal{A}(X, Z) \bigotimes_{\mathbb{k}} T^{\leq N} \mathcal{A}(Z, Y), \\ h_1 \otimes h_2 \otimes \cdots \otimes h_n &\mapsto \sum_{k=0}^n h_1 \otimes \cdots \otimes h_k \bigotimes_{\mathbb{k}} h_{k+1} \otimes \cdots \otimes h_n, \end{aligned}$$

and the counit  $\varepsilon = (T^{\leq N} \mathcal{A}(X, Y) \xrightarrow{\text{pr}_0} T^0 \mathcal{A}(X, Y) \rightarrow \mathbb{k})$ , where the last map is  $\text{id}_{\mathbb{k}}$  if  $X = Y$ , or 0 if  $X \neq Y$  (and  $T^0 \mathcal{A}(X, Y) = 0$ ). We write  $T\mathcal{A}$  instead of  $T^{\leq \infty} \mathcal{A}$ . If  $g : T\mathcal{A} \rightarrow T\mathcal{B}$  is a map of  $\mathbb{k}$ -quivers, then  $g_{ac}$  denotes its matrix coefficient  $T^a \mathcal{A} \xrightarrow{\text{in}_a} T\mathcal{A} \xrightarrow{g} T\mathcal{B} \xrightarrow{\text{pr}_c} T^c \mathcal{B}$ . The matrix coefficient  $g_{a1}$  is abbreviated to  $g_a$ .

**1.2 Definition.** An  $A_N$ -category  $\mathcal{A}$  consists of the following data: a graded  $\mathbb{k}$ -quiver  $\mathcal{A}$ ; a system of  $\mathbb{k}$ -linear maps of degree 1

$$b_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \rightarrow s\mathcal{A}(X_0, X_n), \quad 1 \leq n \leq N,$$

such that for all  $1 \leq k \leq N$

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \rightarrow s\mathcal{A}. \quad (1.2.1)$$

The system  $b_n$  is interpreted as a (1,1)-coderivation  $b : T^{\leq N} s\mathcal{A} \rightarrow T^{\leq N} s\mathcal{A}$  of degree 1 determined by

$$b_{kl} = (b|_{T^k s\mathcal{A}}) \text{pr}_l : T^k s\mathcal{A} \rightarrow T^l s\mathcal{A}, \quad b_{kl} = \sum_{\substack{r+n+t=k \\ r+1+t=l}} 1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}, \quad k, l \leq N,$$

which is a differential.

**1.3 Definition.** A *pointed cocategory homomorphism* consists of the following data:  $A_N$ -categories  $\mathcal{A}$  and  $\mathcal{B}$ , a map  $f : \text{Ob } \mathcal{A} \rightarrow \text{Ob } \mathcal{B}$  and a system of  $\mathbb{k}$ -linear maps of degree 0

$$f_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \rightarrow s\mathcal{B}(X_0 f, X_n f), \quad 1 \leq n \leq N.$$

The above data are equivalent to a cocategory homomorphism  $f : T^{\leq N} s\mathcal{A} \rightarrow T^{\leq N} s\mathcal{B}$  of degree 0 such that

$$f_{01} = (f|_{T^0 s\mathcal{A}}) \text{pr}_1 = 0 : T^0 s\mathcal{A} \rightarrow T^1 s\mathcal{B}, \quad (1.3.1)$$

(this condition was implicitly assumed in [Lyu03, Definition 2.4]). The components of  $f$  are

$$f_{kl} = (f|_{T^k s\mathcal{A}}) \text{pr}_l : T^k s\mathcal{A} \rightarrow T^l s\mathcal{B}, \quad f_{kl} = \sum_{i_1 + \cdots + i_l = k} f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}, \quad (1.3.2)$$

where  $k, l \leq N$ . Indeed, the claim follows from the following diagram, commutative for all  $l \geq 0$ :

$$\begin{array}{ccccc} T^{\leq N} s\mathcal{A} & \xrightarrow{f} & T^{\leq N} s\mathcal{B} & \xrightarrow{\text{pr}_l} & T^l s\mathcal{B} \\ \Delta^{(l)} \downarrow & = & \Delta^{(l)} \downarrow & = & \parallel \\ (T^{\leq N} s\mathcal{A})^{\otimes l} & \xrightarrow{f^{\otimes l}} & (T^{\leq N} s\mathcal{B})^{\otimes l} & \xrightarrow{\text{pr}_1^{\otimes l}} & (s\mathcal{B})^{\otimes l} \end{array}$$

where  $\Delta^{(0)} = \varepsilon$ ,  $\Delta^{(1)} = \text{id}$ ,  $\Delta^{(2)} = \Delta$  and  $\Delta^{(l)}$  means the cut comultiplication, iterated  $l - 1$  times. Notice that condition (1.3.1) can be written as  $f_0 = 0$ .

**1.4 Definition.** An  $A_N$ -functor  $f : \mathcal{A} \rightarrow \mathcal{B}$  is a pointed cocategory homomorphism, which commutes with the differential  $b$ , that is, for all  $1 \leq k \leq N$

$$\sum_{l > 0; i_1 + \cdots + i_l = k} (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}) b_l = \sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k s\mathcal{A} \rightarrow s\mathcal{B}.$$

We are interested mostly in the case  $N = 1$ . Clearly,  $A_1$ -categories are differential graded quivers and  $A_1$ -functors are their morphisms. In the case of one object these reduce to chain complexes and chain maps. The following notion seems interesting even in this case.

**1.5 Definition.** An  $A_N$ -transformation  $r : f \rightarrow g : \mathcal{A} \rightarrow \mathcal{B}$  of degree  $d$  consists of the following data:  $A_N$ -categories  $\mathcal{A}$  and  $\mathcal{B}$ ; pointed cocategory homomorphisms  $f, g : T^{\leq N} s\mathcal{A} \rightarrow T^{\leq N} s\mathcal{B}$  (or  $A_N$ -functors  $f, g : \mathcal{A} \rightarrow \mathcal{B}$ ); a system of  $\mathbb{k}$ -linear maps of degree  $d$

$$r_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \rightarrow s\mathcal{B}(X_0 f, X_n g), \quad 0 \leq n \leq N.$$

To give a system  $r_n$  is equivalent to specifying an  $(f, g)$ -coderivation  $r : T^{\leq N} s\mathcal{A} \rightarrow T^{\leq N+1} s\mathcal{B}$  of degree  $d$

$$\begin{aligned} r_{kl} &= (r|_{T^k s\mathcal{A}}) \text{pr}_l : T^k s\mathcal{A} \rightarrow T^l s\mathcal{B}, \quad k \leq N, l \leq N+1 \\ r_{kl} &= \sum_{\substack{q+1+t=l \\ i_1+\dots+i_q+n+j_1+\dots+j_t=k}} f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_n \otimes g_{j_1} \otimes \dots \otimes g_{j_t}, \end{aligned} \quad (1.5.1)$$

that is, a  $\mathbb{k}$ -quiver morphism  $r$ , satisfying  $r\Delta = \Delta(f \otimes r + r \otimes g)$ . This follows from the commutative diagram

$$\begin{array}{ccccc} T^{\leq N} s\mathcal{A} & \xrightarrow{r} & T^{\leq N+1} s\mathcal{B} & \xrightarrow{\text{pr}_l} & T^l s\mathcal{B} \\ \Delta^{(l)} \downarrow & & \Delta^{(l)} \downarrow & & \parallel \\ (T^{\leq N} s\mathcal{A})^{\otimes l} & \xrightarrow{\sum_{q+1+t=l} f^{\otimes q} \otimes r \otimes g^{\otimes t}} & (T^{\leq N+1} s\mathcal{B})^{\otimes l} & \xrightarrow{\text{pr}_1^{\otimes l}} & (s\mathcal{B})^{\otimes l} \end{array}$$

Let  $\mathcal{A}, \mathcal{B}$  be  $A_N$ -categories, and let  $f^0, f^1, \dots, f^n : T^{\leq N} s\mathcal{A} \rightarrow T^{\leq N} s\mathcal{B}$  be pointed cocategory homomorphisms. Consider coderivations  $r_1, \dots, r_n$  as in

$$f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : T^{\leq N} s\mathcal{A} \rightarrow T^{\leq N} s\mathcal{B}.$$

We construct the following system of  $\mathbb{k}$ -linear maps  $\theta_{kl} : T^k s\mathcal{A} \rightarrow T^l s\mathcal{B}$ ,  $k \leq N, l \leq N+n$  of degree  $\deg r^1 + \dots + \deg r^n$  from these data:

$$\theta_{kl} = \sum f_{i_1^0}^0 \otimes \dots \otimes f_{i_{m_0}^0}^0 \otimes r_{j_1}^1 \otimes f_{i_1^1}^1 \otimes \dots \otimes f_{i_{m_1}^1}^1 \otimes \dots \otimes r_{j_n}^n \otimes f_{i_1^n}^n \otimes \dots \otimes f_{i_{m_n}^n}^n, \quad (1.5.2)$$

where summation is taken over all terms with

$$m_0 + m_1 + \dots + m_n + n = l, \quad i_1^0 + \dots + i_{m_0}^0 + j_1 + i_1^1 + \dots + i_{m_1}^1 + \dots + j_n + i_1^n + \dots + i_{m_n}^n = k.$$

Equivalently, we write

$$\theta_{kl} = \sum_{\substack{m_0+m_1+\dots+m_n+n=l \\ p_0+j_1+p_1+\dots+j_n+p_n=k}} f_{p_0 m_0}^0 \otimes r_{j_1}^1 \otimes f_{p_1 m_1}^1 \otimes \dots \otimes r_{j_n}^n \otimes f_{p_n m_n}^n.$$

The component  $\theta_{kl}$  vanishes unless  $n \leq l \leq k+n$ . If  $n=0$ , then  $\theta_{kl}$  is expansion (1.3.2) of  $f^0$ . If  $n=1$ , then  $\theta_{kl}$  is expansion (1.5.1) of  $r^1$ .

Given an  $A_K$ -category  $\mathcal{A}$  and an  $A_{K+N}$ -category  $\mathcal{B}$ ,  $1 \leq K, N \leq \infty$ , we construct an  $A_N$ -category  $A_K(\mathcal{A}, \mathcal{B})$  out of these. The objects of  $A_K(\mathcal{A}, \mathcal{B})$  are  $A_K$ -functors  $f : \mathcal{A} \rightarrow \mathcal{B}$ . Given two such functors  $f, g : \mathcal{A} \rightarrow \mathcal{B}$  we define the graded  $\mathbb{k}$ -module  $A_K(\mathcal{A}, \mathcal{B})(f, g)$  as the space of all  $A_K$ -transformations  $r : f \rightarrow g$ , namely,

$$\begin{aligned} [A_K(\mathcal{A}, \mathcal{B})(f, g)]^{d+1} \\ = \{r : f \rightarrow g \mid A_K\text{-transformation } r : T^{\leq K} s\mathcal{A} \rightarrow T^{\leq K+1} s\mathcal{B} \text{ has degree } d\}. \end{aligned}$$

The system of differentials  $B_n$ ,  $n \leq N$ , is defined as follows:

$$\begin{aligned}
B_1 : A_K(\mathcal{A}, \mathcal{B})(f, g) &\rightarrow A_K(\mathcal{A}, \mathcal{B})(f, g), \quad r \mapsto (r)B_1 = [r, b] = rb - (-)^r br, \\
[(r)B_1]_k &= \sum_{i_1+\dots+i_q+n+j_1+\dots+j_t=k} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_n \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) b_{q+1+t} \\
&\quad - (-)^r \sum_{\alpha+n+\beta=k} (1^{\otimes \alpha} \otimes b_n \otimes 1^{\otimes \beta}) r_{\alpha+1+\beta}, \quad k \leq K, \\
B_n : A_K(\mathcal{A}, \mathcal{B})(f^0, f^1) \otimes \dots \otimes A_K(\mathcal{A}, \mathcal{B})(f^{n-1}, f^n) &\rightarrow A_K(\mathcal{A}, \mathcal{B})(f^0, f^n), \\
r^1 \otimes \dots \otimes r^n &\mapsto (r^1 \otimes \dots \otimes r^n)B_n, \text{ for } 1 < n \leq N,
\end{aligned}$$

where the last  $A_K$ -transformation is defined by its components:

$$[(r^1 \otimes \dots \otimes r^n)B_n]_k = \sum_{l=n}^{n+k} (r^1 \otimes \dots \otimes r^n) \theta_{kl} b_l, \quad k \leq K.$$

The category of graded  $\mathbb{k}$ -linear quivers admits a symmetric monoidal structure with the tensor product  $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A} \boxtimes \mathcal{B}$ , where  $\text{Ob } \mathcal{A} \boxtimes \mathcal{B} = \text{Ob } \mathcal{A} \times \text{Ob } \mathcal{B}$  and  $(\mathcal{A} \boxtimes \mathcal{B})((X, U), (Y, V)) = \mathcal{A}(X, Y) \otimes_{\mathbb{k}} \mathcal{B}(U, V)$ . The same tensor product was denoted  $\otimes$  in [Lyu03], but we will keep notation  $\mathcal{A} \otimes \mathcal{B}$  only for tensor product from Section 1.1, defined when  $\text{Ob } \mathcal{A} = \text{Ob } \mathcal{B}$ . The two tensor products obey

**Distributivity law.** Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  be graded  $\mathbb{k}$ -linear quivers, such that  $\text{Ob } \mathcal{A} = \text{Ob } \mathcal{B}$  and  $\text{Ob } \mathcal{C} = \text{Ob } \mathcal{D}$ . Then the middle four interchange map  $1 \otimes c \otimes 1$  is an isomorphism of quivers

$$(\mathcal{A} \otimes \mathcal{B}) \boxtimes (\mathcal{C} \otimes \mathcal{D}) \xrightarrow{\sim} (\mathcal{A} \boxtimes \mathcal{C}) \otimes (\mathcal{B} \boxtimes \mathcal{D}), \quad (1.5.3)$$

identity on objects.

Indeed, the both quivers in (1.5.3) have the same set of objects  $R \times S$ , where  $R = \text{Ob } \mathcal{A} = \text{Ob } \mathcal{B}$  and  $S = \text{Ob } \mathcal{C} = \text{Ob } \mathcal{D}$ . Let  $X, Z \in R$  and  $U, W \in S$ . The sets of morphisms from  $(X, U)$  to  $(Z, W)$  are isomorphic via

$$\begin{aligned}
&((\mathcal{A} \otimes \mathcal{B}) \boxtimes (\mathcal{C} \otimes \mathcal{D}))((X, U), (Z, W)) = \\
&(\oplus_{Y \in R} \mathcal{A}(X, Y) \otimes_{\mathbb{k}} \mathcal{B}(Y, Z)) \otimes_{\mathbb{k}} (\oplus_{V \in S} \mathcal{C}(U, V) \otimes_{\mathbb{k}} \mathcal{D}(V, W)) \\
&\quad \downarrow \wr \\
&\oplus_{(Y, V) \in R \times S} \mathcal{A}(X, Y) \otimes_{\mathbb{k}} \mathcal{B}(Y, Z) \otimes_{\mathbb{k}} \mathcal{C}(U, V) \otimes_{\mathbb{k}} \mathcal{D}(V, W) \\
&\quad \downarrow 1 \otimes c \otimes 1 \\
&\oplus_{(Y, V) \in R \times S} \mathcal{A}(X, Y) \otimes_{\mathbb{k}} \mathcal{C}(U, V) \otimes_{\mathbb{k}} \mathcal{B}(Y, Z) \otimes_{\mathbb{k}} \mathcal{D}(V, W) \\
&= ((\mathcal{A} \boxtimes \mathcal{C}) \otimes (\mathcal{B} \boxtimes \mathcal{D}))((X, U), (Z, W)).
\end{aligned}$$

The notion of a pointed cocategory homomorphism extends to the case of several arguments, that is, to degree 0 cocategory homomorphisms  $\psi : T^{\leq L^1} s \mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leq L^q} s \mathcal{C}^q \rightarrow$

$T^{\leq N} s\mathcal{B}$ , where  $N \geq L^1 + \dots + L^q$ . We always assume that  $\psi_{00\dots 0} : T^0 s\mathcal{C}^1 \boxtimes \dots \boxtimes T^0 s\mathcal{C}^q \rightarrow s\mathcal{B}$  vanishes. We call  $\psi$  an  $A$ -functor if it commutes with the differential, that is,

$$(b \boxtimes 1 \boxtimes \dots \boxtimes 1 + 1 \boxtimes b \boxtimes \dots \boxtimes 1 + \dots + 1 \boxtimes 1 \boxtimes \dots \boxtimes b)\psi = \psi b.$$

For example, the map  $\alpha : T^{\leq K} s\mathcal{A} \boxtimes T^{\leq N} sA_K(\mathcal{A}, \mathcal{B}) \rightarrow T^{\leq K+N} s\mathcal{B}$ ,  $a \boxtimes r^1 \otimes \dots \otimes r^n \mapsto a.[(r^1 \otimes \dots \otimes r^n)\theta]$ , is an  $A$ -functor.

**1.6 Proposition** (cf. Proposition 5.5 of [Lyu03]). *Let  $\mathcal{A}$  be an  $A_K$ -category, let  $\mathcal{C}^t$  be an  $A_{L^t}$ -category for  $1 \leq t \leq q$ , and let  $\mathcal{B}$  be an  $A_N$ -category, where  $N \geq K + L^1 + \dots + L^q$ . For any  $A$ -functor  $\phi : T^{\leq K} s\mathcal{A} \boxtimes T^{\leq L^1} s\mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leq L^q} s\mathcal{C}^q \rightarrow T^{\leq N} s\mathcal{B}$  there is a unique  $A$ -functor  $\psi : T^{\leq L^1} s\mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leq L^q} s\mathcal{C}^q \rightarrow T^{\leq N-K} sA_K(\mathcal{A}, \mathcal{B})$ , such that*

$$\phi = (T^{\leq K} s\mathcal{A} \boxtimes T^{\leq L^1} s\mathcal{C}^1 \boxtimes \dots \boxtimes T^{\leq L^q} s\mathcal{C}^q \xrightarrow{1 \boxtimes \psi} T^{\leq K} s\mathcal{A} \boxtimes T^{\leq N-K} sA_K(\mathcal{A}, \mathcal{B}) \xrightarrow{\alpha} T^{\leq N} s\mathcal{B}).$$

Let  $\mathcal{A}$  be an  $A_N$ -category, let  $\mathcal{B}$  be an  $A_{N+K}$ -category, and let  $\mathcal{C}$  be an  $A_{N+K+L}$ -category. The above proposition implies the existence of an  $A$ -functor (cf. [Lyu03, Proposition 4.1])

$$M : T^{\leq K} sA_N(\mathcal{A}, \mathcal{B}) \boxtimes T^{\leq L} sA_{N+K}(\mathcal{B}, \mathcal{C}) \rightarrow T^{\leq K+L} sA_N(\mathcal{A}, \mathcal{C}),$$

in particular,  $(1 \boxtimes B + B \boxtimes 1)M = MB$ . It has the components

$$M_{nm} = M|_{T^n \boxtimes T^m} \text{pr}_1 : T^n sA_N(\mathcal{A}, \mathcal{B}) \boxtimes T^m sA_{N+K}(\mathcal{B}, \mathcal{C}) \rightarrow sA_N(\mathcal{A}, \mathcal{C}),$$

$n \leq K$ ,  $m \leq L$ . We have  $M_{00} = 0$  and  $M_{nm} = 0$  for  $m > 1$ . If  $m = 0$  and  $n$  is positive,  $M_{n0}$  is given by the formula:

$$\begin{aligned} M_{n0} : sA_N(\mathcal{A}, \mathcal{B})(f^0, f^1) \otimes \dots \otimes sA_N(\mathcal{A}, \mathcal{B})(f^{n-1}, f^n) \boxtimes \mathbb{k}_{g^0} &\rightarrow sA_N(\mathcal{A}, \mathcal{C})(f^0 g^0, f^n g^0), \\ r^1 \otimes \dots \otimes r^n \boxtimes 1 &\mapsto (r^1 \otimes \dots \otimes r^n | g^0) M_{n0}, \end{aligned}$$

$$[(r^1 \otimes \dots \otimes r^n | g^0) M_{n0}]_k = \sum_{l=n}^{n+k} (r^1 \otimes \dots \otimes r^n) \theta_{kl} g_l^0, \quad k \leq N,$$

where  $|$  separates the arguments in place of  $\boxtimes$ . If  $m = 1$ , then  $M_{n1}$  is given by the formula:

$$\begin{aligned} M_{n1} : sA_N(\mathcal{A}, \mathcal{B})(f^0, f^1) \otimes \dots \otimes sA_N(\mathcal{A}, \mathcal{B})(f^{n-1}, f^n) \boxtimes sA_{N+K}(\mathcal{B}, \mathcal{C})(g^0, g^1) \\ \rightarrow sA_N(\mathcal{A}, \mathcal{C})(f^0 g^0, f^n g^1), \quad r^1 \otimes \dots \otimes r^n \boxtimes t^1 \mapsto (r^1 \otimes \dots \otimes r^n \boxtimes t^1) M_{n1}, \end{aligned}$$

$$[(r^1 \otimes \dots \otimes r^n \boxtimes t^1) M_{n1}]_k = \sum_{l=n}^{n+k} (r^1 \otimes \dots \otimes r^n) \theta_{kl} t_l^1, \quad k \leq N.$$

Note that equations

$$[(r^1 \otimes \dots \otimes r^n) B_n]_k = [(r^1 \otimes \dots \otimes r^n \boxtimes b) M_{n1}]_k - (-)^{r^1 + \dots + r^n} [(b \boxtimes r^1 \otimes \dots \otimes r^n) M_{1n}]_k$$

imply that

$$(r^1 \otimes \cdots \otimes r^n)B_n = (r^1 \otimes \cdots \otimes r^n \boxtimes b)M_{n1} - (-)^{r^1+\cdots+r^n}(b \boxtimes r^1 \otimes \cdots \otimes r^n)M_{1n},$$

$$B = (1 \boxtimes b)M - (b \boxtimes 1)M : \text{id} \rightarrow \text{id} : A_N(\mathcal{A}, \mathcal{B}) \rightarrow A_N(\mathcal{A}, \mathcal{B}).$$

Proposition 1.6 implies the existence of a unique  $A_L$ -functor

$$A_N(\mathcal{A}, -) : A_{N+K}(\mathcal{B}, \mathcal{C}) \rightarrow A_K(A_N(\mathcal{A}, \mathcal{B}), A_N(\mathcal{A}, \mathcal{C})),$$

such that

$$M = [T^{\leq K} sA_N(\mathcal{A}, \mathcal{B}) \boxtimes T^{\leq L} sA_{N+K}(\mathcal{B}, \mathcal{C}) \xrightarrow{1 \boxtimes A_N(\mathcal{A}, -)} T^{\leq K} sA_N(\mathcal{A}, \mathcal{B}) \boxtimes T^{\leq L} sA_K(A_N(\mathcal{A}, \mathcal{B}), A_N(\mathcal{A}, \mathcal{C})) \xrightarrow{\alpha} T^{\leq K+L} sA_N(\mathcal{A}, \mathcal{C})].$$

The  $A_L$ -functor  $A_N(\mathcal{A}, -)$  is strict, cf. [Lyu03, Proposition 6.2].

Let  $\mathcal{A}$  be an  $A_N$ -category, and let  $\mathcal{B}$  be a unital  $A_\infty$ -category with a unit transformation  $\mathbf{i}^{\mathcal{B}}$ . Then  $A_N(\mathcal{A}, \mathcal{B})$  is a unital  $A_\infty$ -category with the unit transformation  $(1 \boxtimes \mathbf{i}^{\mathcal{B}})M$  (cf. [Lyu03, Proposition 7.7]). The unit element for an object  $f \in \text{Ob } A_N(\mathcal{A}, \mathcal{B})$  is  $f \mathbf{i}_0^{A_N(\mathcal{A}, \mathcal{B})} : \mathbb{k} \rightarrow (sA_N)^{-1}(\mathcal{A}, \mathcal{B}), 1 \mapsto f \mathbf{i}^{\mathcal{B}}$ .

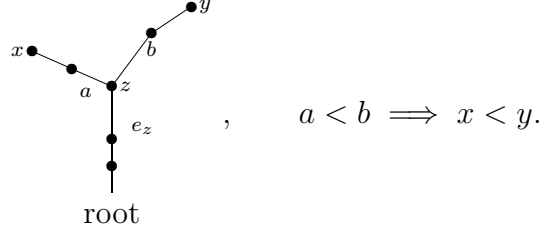
When  $\mathcal{A}$  is an  $A_K$ -category and  $N < K$ , we may forget part of its structure and view  $\mathcal{A}$  as an  $A_N$ -category. If furthermore,  $\mathcal{B}$  is an  $A_{K+L}$ -category, we have the restriction strict  $A_L$ -functor  $\text{restr}_{K,N} : A_K(\mathcal{A}, \mathcal{B}) \rightarrow A_N(\mathcal{A}, \mathcal{B})$ . To prove the results mentioned above, we notice that they are restrictions of their  $A_\infty$ -analogs to finite  $N$ . Since the proofs of  $A_\infty$ -results are obtained in [Lyu03] by induction, an inspection shows that the proofs of the above  $A_N$ -statements are obtained as a byproduct.

**1.7. Trees.** Since the notions related to trees might be interpreted with some variations, we give precise definitions and fix notation. A *tree* is a non-empty connected graph without cycles. A vertex which belongs to only one edge is called *external*, other vertices are *internal*. A *plane tree* is a tree equipped for each internal vertex  $v$  with a cyclic ordering of the set  $E_v$  of edges, adjacent to  $v$ . Plane trees can be drawn on an oriented plane in a unique way (up to an ambient isotopy) so that the cyclic ordering of each  $E_v$  agrees with the orientation of the plane. An external vertex distinct from the root is called *input vertex*.

A *rooted tree* is a tree with a distinguished external vertex, called *root*. The set of vertices  $V(t)$  of a rooted tree  $t$  has a canonical ordering:  $x \preccurlyeq y$  iff the minimal path connecting the root with  $y$  contains  $x$ . A *linearly ordered tree* is a rooted tree  $t$  equipped with a linear order  $\leq$  of the set of internal vertices  $IV(t)$ , such that for all internal vertices  $x, y$  the relation  $x \preccurlyeq y$  implies  $x \leq y$ . For each vertex  $v \in V(t) - \{\text{root}\}$  of a rooted tree, the set  $E_v$  has a distinguished element  $e_v$  – the beginning of a minimal path from  $v$  to the root. Therefore, for each vertex  $v \in V(t) - \{\text{root}\}$  of a rooted plane tree, the set  $E_v$  admits a unique linear order  $<$ , for which  $e_v$  is minimal and the induced cyclic order is the given one. An internal vertex  $v$  has degree  $d$ , if  $\text{Card}(E_v) = d + 1$ .



For any  $y \in V(t)$  let  $P_y = \{x \in V(t) \mid x \preccurlyeq y\}$ . With each plane rooted tree  $t$  is associated a linearly ordered tree  $t_{\leq} = (t, \leq)$  as follows. If  $x, y \in IV(t)$  are such that  $x \not\preccurlyeq y$  and  $y \not\preccurlyeq x$ , then  $P_x \cap P_y = P_z$  for a unique  $z \in IV(t)$ , distinct from  $x$  and  $y$ . Let  $a \in E_z - \{e_z\}$  (resp.  $b \in E_z - \{e_z\}$ ) be the beginning of the minimal path connecting  $z$  and  $x$  (resp.  $y$ ). If  $a < b$ , we set  $x < y$ . Graphically we  $<$ -order the internal vertices by height. Thus, an internal vertex  $x$  on the left is depicted lower than a  $\preccurlyeq$ -incomparable internal vertex  $y$  on the right:



A *forest* is a sequence of plane rooted trees. Concatenation of forests is denoted  $\sqcup$ . The vertical composition  $F_1 \cdot F_2$  of forests  $F_1, F_2$  is well-defined if the sum of lengths of sequences  $F_1$  and  $F_2$  equals the number of external vertices of  $F_2$ . These operations allow to construct any tree from elementary ones

$$1 = \begin{array}{c} | \\ | \\ | \end{array}, \quad \text{and} \quad \mathbf{t}_k = \begin{array}{c} \cdots \\ \diagup \quad \diagdown \\ | \end{array} \quad (k \text{ input vertices}).$$

Namely, any linearly ordered tree  $(t, \leq)$  has a unique presentation of the form

$$(t, \leq) = (1^{\sqcup \alpha_1} \sqcup \mathbf{t}_{k_1} \sqcup 1^{\sqcup \beta_1}) \cdot (1^{\sqcup \alpha_2} \sqcup \mathbf{t}_{k_2} \sqcup 1^{\sqcup \beta_2}) \cdot \dots \cdot \mathbf{t}_{k_N}, \quad (1.7.1)$$

where  $N = |t| \stackrel{\text{def}}{=} \text{Card}(IV(t))$  is the number of internal vertices. Here

$$1^{\sqcup \alpha} \sqcup \mathbf{t}_k \sqcup 1^{\sqcup \beta} = \begin{array}{c} \overbrace{\begin{array}{|c|c|c|} \hline | & \dots & | \\ \hline \end{array}}^{\alpha} \quad \begin{array}{c} \overbrace{\begin{array}{c} \cdots \\ \diagup \quad \diagdown \\ | \end{array}}^k \\ \end{array} \quad \begin{array}{c} \overbrace{\begin{array}{|c|c|c|} \hline | & \dots & | \\ \hline \end{array}}^{\beta} \end{array}.$$

In (1.7.1) the highest vertex is indexed by 1, the lowest – by  $N$ .

## 2. Properties of free $A_\infty$ -categories

**2.1. Construction of a free  $A_\infty$ -category.** The category  $\text{strict}A_\infty$  has  $A_\infty$ -categories as objects and strict  $A_\infty$ -functors as morphisms. There is a functor  $\mathcal{U} : \text{strict}A_\infty \rightarrow A_1$ ,  $\mathcal{A} \mapsto (\mathcal{A}, m_1)$  which sends an  $A_\infty$ -category to the underlying differential graded  $\mathbb{k}$ -quiver, forgetting all higher multiplications. Following Kontsevich and Soibelman [KS02] we are going to prove that  $\mathcal{U}$  has a left adjoint functor  $\mathcal{F} : A_1 \rightarrow \text{strict}A_\infty$ ,  $\mathcal{Q} \mapsto \mathcal{F}\mathcal{Q}$ . The  $A_\infty$ -category  $\mathcal{F}\mathcal{Q}$  is called free. Below we describe its structure for an arbitrary differential graded  $\mathbb{k}$ -quiver  $\mathcal{Q}$ . We shall work with its shift  $(s\mathcal{Q}, d)$ .

Let us define an  $A_\infty$ -category  $\mathcal{FQ}$  via the following data. The class of objects  $\text{Ob } \mathcal{FQ}$  is  $\text{Ob } \mathcal{Q}$ . The  $\mathbb{Z}$ -graded  $\mathbb{k}$ -modules of morphisms between  $X, Y \in \text{Ob } \mathcal{Q}$  are

$$\begin{aligned} s\mathcal{FQ}(X, Y) &= \bigoplus_{n \geq 1} \bigoplus_{t \in \mathcal{T}_{\geq 2}^n} s\mathcal{F}_t\mathcal{Q}(X, Y), \\ s\mathcal{F}_t\mathcal{Q}(X, Y) &= \bigoplus_{\substack{X_0=X, X_n=Y \\ X_0, \dots, X_n \in \text{Ob } \mathcal{Q}}} s\mathcal{Q}(X_0, X_1) \otimes \cdots \otimes s\mathcal{Q}(X_{n-1}, X_n)[-|t|], \end{aligned}$$

where  $\mathcal{T}_{\geq 2}^n$  is the class of plane rooted trees with  $n+1$  external vertices, such that  $\text{Card}(E_v) \geq 3$  for all  $v \in IV(t)$ . We use the following convention: if  $M, N$  are (differential) graded  $\mathbb{k}$ -modules, then<sup>1</sup>

$$\begin{aligned} (M \otimes N)[k] &= M \otimes (N[k]), \\ (M \otimes N \xrightarrow{s^k} (M \otimes N)[k]) &= (M \otimes N \xrightarrow{1 \otimes s^k} M \otimes (N[k])). \end{aligned}$$

The quiver  $\mathcal{FQ}$  is equipped with the following operations. For  $k > 1$  the operation  $b_k$  is a direct sum of maps

$$b_k = s^{|t_1|} \otimes \cdots \otimes s^{|t_{k-1}|} \otimes s^{|t_k| - |t|} : s\mathcal{F}_{t_1}\mathcal{Q}(Y_0, Y_1) \otimes \cdots \otimes s\mathcal{F}_{t_k}\mathcal{Q}(Y_{k-1}, Y_k) \rightarrow s\mathcal{F}_t\mathcal{Q}(Y_0, Y_k), \quad (2.1.1)$$

where  $t = (t_1 \sqcup \cdots \sqcup t_k) \cdot t_k$ . In particular,  $|t| = |t_1| + \cdots + |t_k| + 1$ . The operation  $b_1$  restricted to  $s\mathcal{F}_t\mathcal{Q}$  is

$$b_1 = d \oplus (-1)^{\beta(t')} s^{-1} : s\mathcal{F}_t\mathcal{Q}(X, Y) \rightarrow s\mathcal{F}_t\mathcal{Q}(X, Y) \oplus \bigoplus_{t'=t+\text{edge}} s\mathcal{F}_{t'}\mathcal{Q}(X, Y), \quad (2.1.2)$$

where the sum extends over all trees  $t' \in \mathcal{T}_{\geq 2}^n$  with a distinguished edge  $e$ , such that contracting  $e$  we get  $t$  from  $t'$ . The sign is determined by

$$\beta(t') = \beta(t', e) = 1 + h(\text{highest vertex of } e),$$

where an isomorphism of ordered sets

$$h : IV(t'_<) \xrightarrow{\sim} [1, |t'|] \cap \mathbb{Z}$$

is simply the height of a vertex in the linearly ordered tree  $t'_<$ , canonically associated with  $t'$ . In (2.1.2)  $d$  means  $d \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes d \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes d$ , where the last  $d$  is  $d_{s\mathcal{Q}[-|t|]} = (-1)^{|t|} s^{|t|} \cdot d_{s\mathcal{Q}} \cdot s^{-|t|}$ , as usual. According to our conventions,  $s^{-1}$  in (2.1.2) means  $1^{\otimes n-1} \otimes s^{-1}$ .

**2.2 Proposition.**  $\mathcal{FQ}$  is an  $A_\infty$ -category.

*Proof.* First we prove that  $b_1^2 = 0$  on  $s\mathcal{F}_t\mathcal{Q}$ . Indeed,

$$\begin{aligned} b_1^2 &= d^2 \oplus (-1)^{\beta(t', e)} (s^{-1}d + ds^{-1}) \oplus [(-1)^{\beta(t'_1, e_1) + \beta(t'', e_2)} + (-1)^{\beta(t'_2, e_2) + \beta(t'', e_1)}] s^{-2} : \\ &\quad s\mathcal{F}_t\mathcal{Q} \rightarrow s\mathcal{F}_t\mathcal{Q} \oplus \bigoplus_{t'=t+e} s\mathcal{F}_{t'}\mathcal{Q} \oplus \bigoplus_{t''=t+e_1+e_2} s\mathcal{F}_{t''}\mathcal{Q}, \end{aligned}$$

---

<sup>1</sup>Another gauge choice  $(M \otimes N)[1] = M[1] \otimes N$ ,  $s = s \otimes 1$  seems less convenient.

where  $t''$  contains two distinguished edges  $e_1, e_2$ , contraction along which gives  $t$ ;  $t'_2$  is  $t''$  contracted along  $e_1$ , and  $t'_1$  is  $t''$  contracted along  $e_2$ . We may assume that highest vertex of  $e_1$  is lower than highest vertex of  $e_2$  in  $t''$ . Then  $\beta(t'_1, e_1) = \beta(t'', e_1)$  and  $\beta(t'', e_2) = \beta(t'_2, e_2) + 1$ , hence,

$$(-1)^{\beta(t'_1, e_1) + \beta(t'', e_2)} + (-1)^{\beta(t'_2, e_2) + \beta(t'', e_1)} = 0.$$

Obviously,  $d^2 = 0$  and  $s^{-1}d + ds^{-1} = 0$ , hence,  $b_1^2 = 0$ .

Let us prove for each  $n > 1$  that

$$b_n b_1 + \sum_{p=1}^n (1^{\otimes p-1} \otimes b_1 \otimes 1^{\otimes n-p}) b_n + \sum_{\substack{\alpha+k+\beta=n \\ \alpha+\beta>0}}^{k>1} (1^{\otimes \alpha} \otimes b_k \otimes 1^{\otimes \beta}) b_{\alpha+1+\beta} = 0 :$$

$$s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \rightarrow s\mathcal{F}_t\mathcal{Q} \oplus \bigoplus_{p, t'_p} s\mathcal{F}_{t'_p}\mathcal{Q} \oplus \bigoplus_{\substack{k>1, t'' \\ \alpha+k+\beta=n \\ \alpha+\beta>0}} s\mathcal{F}_{t''}\mathcal{Q},$$

where

$$t = (t_1 \sqcup \cdots \sqcup t_n) \cdot \mathbf{t}_n = \begin{array}{c} \boxed{t_1} \quad \boxed{t_2} \quad \cdots \quad \boxed{t_{n-1}} \quad \boxed{t_n} \\ \diagdown \quad \diagup \quad \cdots \quad \diagdown \quad \diagup \\ \bullet \\ \downarrow \end{array}, \quad (2.2.1)$$

$$t' = (t_1 \sqcup \cdots \sqcup t'_p \sqcup \cdots \sqcup t_n) \cdot \mathbf{t}_n = \begin{array}{c} \boxed{t_1} \quad \boxed{t'_p} \quad \boxed{t_n} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \\ \downarrow \end{array},$$

$$\begin{aligned} t'' &= (t_1 \sqcup \cdots \sqcup t_n) \cdot (1^{\sqcup \alpha} \sqcup \mathbf{t}_k \sqcup 1^{\sqcup \beta}) \cdot \mathbf{t}_{\alpha+1+\beta} \\ &= \begin{array}{c} \boxed{t_1} \quad \boxed{t_\alpha} \quad \boxed{t_{\alpha+1}} \quad \boxed{t_{\alpha+k}} \quad \boxed{t_{\alpha+k+1}} \quad \boxed{t_n} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \bullet \\ \downarrow \end{array}, \end{aligned}$$

and contraction of  $t'_p$  along distinguished edge  $e_p$  gives  $t_p$ . According to the three types of summands in the target, the required equation follows from anticommutativity of the following three diagrams:

$$\begin{array}{ccc} s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow[b_n]{s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}} & s\mathcal{F}_t\mathcal{Q} \\ \downarrow 1^{\otimes n-1} \otimes d + \cdots + d \otimes 1^{\otimes n-1} & \quad \quad \quad \downarrow d & \\ s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow[b_n]{s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}} & s\mathcal{F}_t\mathcal{Q} \end{array}$$

that is,

$$\begin{aligned} & (1^{\otimes n-1} \otimes d + \cdots + d \otimes 1^{\otimes n-1}) (s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}) \\ & + (s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}) (1^{\otimes n-1} \otimes d + \cdots + d \otimes 1^{\otimes n-1}) = 0; \end{aligned}$$

$$\begin{array}{ccc}
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_p}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow{s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}} & s\mathcal{F}_t\mathcal{Q} \\
\downarrow (-)^{\beta(t'_p)} 1^{\otimes p-1} \otimes s^{-1} \otimes 1^{\otimes n-p} & & \downarrow (-)^{1+|t_1|+\cdots+|t_{p-1}|+\beta(t'_p)} s^{-1} \\
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t'_p}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow{s^{|t_1|} \otimes \cdots \otimes s^{|t_{p-1}|} \otimes s^{|t_p|+1} \otimes s^{|t_{p+1}|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|-1}} & s\mathcal{F}_{t'}\mathcal{Q}
\end{array}$$

that is,

$$\begin{aligned}
& (-1)^{\beta(t'_p)} (1^{\otimes p-1} \otimes s^{-1} \otimes 1^{\otimes n-p}) (s^{|t_1|} \otimes \cdots \otimes s^{|t_{p-1}|} \otimes s^{|t_p|+1} \otimes s^{|t_{p+1}|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|-1}) \\
& + (s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}) (-1)^{1+|t_1|+\cdots+|t_{p-1}|+\beta(t'_p)} (1^{\otimes n-1} \otimes s^{-1}) = 0,
\end{aligned}$$

in the particular case  $p = n$  it holds as well;

$$\begin{array}{ccc}
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow{s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}} & s\mathcal{F}_t\mathcal{Q} \\
\downarrow 1^{\otimes \alpha} \otimes s^{|t_{\alpha+1}|} \otimes \cdots \otimes s^{|t_{\alpha+k-1}|} \otimes s^{-|t_{\alpha+1}|-\cdots-|t_{\alpha+k-1}|-1} \otimes 1^{\otimes \beta} & & \downarrow (-)^{1+|t_1|+\cdots+|t_{\alpha}|} s^{-1} \\
s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_{\alpha}}\mathcal{Q} \otimes s\mathcal{F}_{\hat{t}}\mathcal{Q} \otimes s\mathcal{F}_{t_{\alpha+k+1}}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow{s^{|t_1|} \otimes \cdots \otimes s^{|t_{\alpha}|} \otimes s^{|t_{\alpha+k+1}|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|-1}} & s\mathcal{F}_{t''}\mathcal{Q}
\end{array}$$

where  $\hat{t} = (t_{\alpha+1} \sqcup \cdots \sqcup t_{\alpha+k}) \cdot t_k$ , that is,

$$\begin{aligned}
& (1^{\otimes \alpha} \otimes s^{|t_{\alpha+1}|} \otimes \cdots \otimes s^{|t_{\alpha+k-1}|} \otimes s^{-|t_{\alpha+1}|-\cdots-|t_{\alpha+k-1}|-1} \otimes 1^{\otimes \beta}). \\
& (s^{|t_1|} \otimes \cdots \otimes s^{|t_{\alpha}|} \otimes 1^{\otimes k-1} \otimes s^{|t_{\alpha+1}|+\cdots+|t_{\alpha+k}|+1} \otimes s^{|t_{\alpha+k+1}|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|-1}) \\
& + (s^{|t_1|} \otimes \cdots \otimes s^{|t_{n-1}|} \otimes s^{|t_n|-|t|}) (-1)^{1+|t_1|+\cdots+|t_{\alpha}|} (1^{\otimes n-1} \otimes s^{-1}) = 0,
\end{aligned}$$

in the particular case  $\beta = 0$  it holds as well.

Therefore,  $\mathcal{FQ}$  is an  $A_{\infty}$ -category.  $\square$

Let us establish a property of free  $A_{\infty}$ -categories, which explains why they are called free.

**2.3 Proposition** ( $A_{\infty}$ -functors from a free  $A_{\infty}$ -category). *Let  $\mathcal{Q}$  be a differential graded quiver, and let  $\mathcal{A}$  be an  $A_{\infty}$ -category. Let  $f_1 : s\mathcal{Q} \rightarrow (s\mathcal{A}, b_1)$  be a chain morphism of differential graded quivers with the underlying mapping of objects  $\text{Ob } f : \text{Ob } \mathcal{Q} \rightarrow \text{Ob } \mathcal{A}$ . Suppose given  $\mathbb{k}$ -quiver morphisms  $f_k : T^k s\mathcal{FQ} \rightarrow s\mathcal{A}$  of degree 0 with the same underlying map  $\text{Ob } f$  for all  $k > 1$ . Then there exists a unique extension of  $f_1$  to a quiver morphism  $f_1 : s\mathcal{FQ} \rightarrow s\mathcal{A}$  such that  $(f_1, f_2, \dots)$  are components of an  $A_{\infty}$ -functor  $f : \mathcal{FQ} \rightarrow \mathcal{A}$ .*

*Proof.* For each  $n > 1$  we have to satisfy the equation

$$b_n f_1 = \sum_{i_1+\cdots+i_l=n} (f_{i_1} \otimes \cdots \otimes f_{i_l}) b_l - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes \alpha} \otimes b_k \otimes 1^{\otimes \beta}) f_{\alpha+1+\beta} : T^n s\mathcal{FQ} \rightarrow s\mathcal{A}. \quad (2.3.1)$$

It is used to define recursively  $f_1$  on  $s\mathcal{F}\mathcal{Q}$ . Suppose that  $t_1, \dots, t_n$  are trees,  $n > 1$ , and  $f_1 : s\mathcal{F}_{t_i}\mathcal{Q} \rightarrow s\mathcal{A}$  is already defined for all  $1 \leq i \leq n$ . Since

$$b_n = s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n| - |t|} : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \rightarrow s\mathcal{F}_t\mathcal{Q}$$

is invertible for  $t = (t_1 \sqcup \dots \sqcup t_n) \cdot \mathbf{t}_n$ , formula (2.3.1) determines  $f_1 : s\mathcal{F}_t\mathcal{Q} \rightarrow s\mathcal{A}$  uniquely as

$$f_1 = (s\mathcal{F}_t\mathcal{Q} \xrightarrow{b_n^{-1}} s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \xrightarrow{\sum (f_{i_1} \otimes \dots \otimes f_{i_l}) b_l - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) f_{\alpha+1+\beta}} s\mathcal{A}).$$

This proves uniqueness of the extension of  $f_1$ .

Let us prove that the cocategory homomorphism  $f$  with so defined components  $(f_1, f_2, \dots)$  is an  $A_\infty$ -functor. Equations (2.3.1) are satisfied by construction of  $f_1$ . So it remains to prove that  $f_1$  is a chain map. Equation  $f_1 b_1 = b_1 f_1$  holds on  $s\mathcal{F}_|\mathcal{Q}$  by assumption. We are going to prove by induction on  $|t|$  that it holds on  $s\mathcal{F}_t\mathcal{Q}$ . Considering  $t = (t_1 \sqcup \dots \sqcup t_n) \cdot \mathbf{t}_n$ ,  $n > 1$ , we assume that  $f_1 b_1 = b_1 f_1 : s\mathcal{F}_{t'}\mathcal{Q} \rightarrow s\mathcal{A}$  for all trees  $t'$  with  $|t'| < |t|$ . To prove that  $f_1 b_1 = b_1 f_1 : s\mathcal{F}_t\mathcal{Q} \rightarrow s\mathcal{A}$  it suffices to show that  $b_n f_1 b_1 = b_n b_1 f_1$  for all  $n > 1$  due to invertibility of  $b_n$ . Using (2.3.1) and the equation  $b^2 \text{pr}_1 = 0$  we find

$$\begin{aligned} b_n f_1 b_1 - b_n b_1 f_1 &= \sum_{i_1 + \dots + i_l = n} (f_{i_1} \otimes \dots \otimes f_{i_l}) b_l b_1 - \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) f_{\alpha+1+\beta} b_1 \\ &\quad + \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) b_{\alpha+1+\beta} f_1 \\ &= - \sum_{i_1 + \dots + i_l = n} (f_{i_1} \otimes \dots \otimes f_{i_l}) \sum_{\gamma+p+\delta=l}^{\gamma+\delta>0} (1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta}) b_{\gamma+1+\delta} \\ &\quad + \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \left[ \sum_{j_1 + \dots + j_r = \alpha+1+\beta}^{r>1} (f_{j_1} \otimes \dots \otimes f_{j_r}) b_r \right. \\ &\quad \left. - \sum_{\gamma+p+\delta=\alpha+1+\beta}^{\gamma+\delta>0} (1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta}) f_{\gamma+1+\delta} \right] \\ &= \sum_{r>1} \left[ \sum_{\alpha+k+\beta=n}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \sum_{j_1 + \dots + j_r = \alpha+1+\beta} f_{j_1} \otimes \dots \otimes f_{j_r} \right. \\ &\quad \left. - \sum_{i_1 + \dots + i_l = n} (f_{i_1} \otimes \dots \otimes f_{i_l}) \sum_{\substack{\gamma+p+\delta=l \\ \gamma+1+\delta=r}} 1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta} \right] b_r \\ &\quad - \sum_{r>1} \left[ \sum_{\substack{\alpha+k+\beta=n \\ \alpha+\beta>0}} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \sum_{\substack{\gamma+p+\delta=\alpha+1+\beta \\ \gamma+1+\delta=r}} 1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta} \right] f_r. \end{aligned}$$

Let us show that the expressions in square brackets vanish. The first one is the matrix coefficient  $bf - fb : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \rightarrow T^r s\mathcal{A}$ . Indeed, for  $r > 1$  the inequality  $r \leq j_1 + \cdots + j_r = \alpha + 1 + \beta$  automatically implies that  $\alpha + \beta > 0$ , so this condition can be omitted. Using the induction hypothesis one can transform the left hand side of equation

$$\begin{aligned} & \sum_{\alpha+k+\beta=n} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \sum_{j_1+\cdots+j_r=\alpha+1+\beta} f_{j_1} \otimes \cdots \otimes f_{j_r} \\ &= \sum_{i_1+\cdots+i_l=n} (f_{i_1} \otimes \cdots \otimes f_{i_l}) \sum_{\substack{\gamma+p+\delta=l \\ \gamma+1+\delta=r}} 1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta} : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_n}\mathcal{Q} \rightarrow T^r s\mathcal{A} \end{aligned}$$

into the right hand side for all  $n, r \geq 1$ .

The second expression

$$\sum_{\substack{\alpha+k+\beta=n \\ \alpha+\beta>0}} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) \sum_{\substack{\gamma+p+\delta=\alpha+1+\beta \\ \gamma+1+\delta=r}} 1^{\otimes\gamma} \otimes b_p \otimes 1^{\otimes\delta} \quad (2.3.2)$$

is the matrix coefficient

$$(b - b \operatorname{pr}_1) b \operatorname{pr}_r : T^n s\mathcal{F}\mathcal{Q} \rightarrow T^r s\mathcal{F}\mathcal{Q}$$

of the endomorphism  $(b - b \operatorname{pr}_1) b : T s\mathcal{F}\mathcal{Q} \rightarrow T s\mathcal{F}\mathcal{Q}$ . However,

$$(b - b \operatorname{pr}_1) b \operatorname{pr}_r = b^2 \operatorname{pr}_r - b \operatorname{pr}_1 b \operatorname{pr}_r = -b \operatorname{pr}_1 b \operatorname{pr}_1 \operatorname{pr}_r = 0$$

for  $r > 1$ , because  $\operatorname{pr}_1 b = \operatorname{pr}_1 b \operatorname{pr}_1$ . Therefore, (2.3.2) vanishes and equation  $b_n f_1 b_1 = b_n b_1 f_1$  is proven.  $\square$

Let  $\operatorname{strict} A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \subset A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A})$  be a full  $A_\infty$ -subcategory, whose objects are strict  $A_\infty$ -functors. Recall that  $\operatorname{Ob} A_1(\mathcal{Q}, \mathcal{A})$  is the set of chain morphisms  $\mathcal{Q} \rightarrow \mathcal{A}$  of differential graded quivers.

**2.4 Corollary.** *A chain morphism  $f : \mathcal{Q} \rightarrow \mathcal{A}$  admits a unique extension to a strict  $A_\infty$ -functor  $\hat{f} : \mathcal{F}\mathcal{Q} \rightarrow \mathcal{A}$ . The maps  $f \mapsto \hat{f}$  and*

$$\operatorname{restr} : \operatorname{Ob} \operatorname{strict} A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \rightarrow \operatorname{Ob} A_1(\mathcal{Q}, \mathcal{A}), \quad g \mapsto (\operatorname{Ob} g, g_1|_{s\mathcal{Q}})$$

*are inverse to each other.*

Indeed, strict  $A_\infty$ -functors  $g$  are distinguished by conditions  $g_k = 0$  for  $k > 1$ .

We may view  $\operatorname{strict} A_\infty$  as a category, whose objects are  $A_\infty$ -categories and morphisms are strict  $A_\infty$ -functors. We may also view  $A_1$  as a category consisting of differential graded quivers and their morphisms. There is a functor  $\mathcal{U} : \operatorname{strict} A_\infty \rightarrow A_1$ ,  $\mathcal{A} \mapsto (\mathcal{A}, m_1)$ , which sends an  $A_\infty$ -category to the underlying differential graded  $\mathbb{k}$ -quiver, forgetting all higher multiplications. The restriction map

$$\operatorname{restr} : \operatorname{strict} A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{U}\mathcal{A}), \quad g \mapsto (\operatorname{Ob} g, g_1|_{s\mathcal{Q}}) \quad (2.4.1)$$

is functorial in  $\mathcal{A}$ .

**2.5 Corollary.** *There is a functor  $\mathcal{F} : A_1 \rightarrow \operatorname{strict} A_\infty$ ,  $\mathcal{Q} \mapsto \mathcal{F}\mathcal{Q}$ , left adjoint to  $\mathcal{U}$ .*

**2.6. Explicit formula for the constructed strict  $A_\infty$ -functor.** Let us obtain a more explicit formula for  $\widehat{f}_1|_{s\mathcal{F}_t\mathcal{Q}}$ . We define  $\widehat{f}_1$  for  $t = (t_1 \sqcup \dots \sqcup t_n) \cdot \mathbf{t}_n$  recursively by a commutative diagram

$$\begin{array}{ccc} s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_n}\mathcal{Q} & \xrightarrow{s^{|t_1|} \otimes \dots \otimes s^{|t_{n-1}|} \otimes s^{|t_n| - |t|}} & s\mathcal{F}_t\mathcal{Q} \\ \widehat{f}_1^{\otimes n} \downarrow & \stackrel{\text{def}}{=} & \downarrow \widehat{f}_1 \\ s\mathcal{A} \otimes \dots \otimes s\mathcal{A} & \xrightarrow{b_n} & s\mathcal{A} \end{array}$$

Notice that the top map is invertible. Here  $n, t_1, \dots, t_n$  are uniquely determined by decomposition (2.2.1) of  $t$ .

Let  $(t, \leq)$  be a linearly ordered tree with the underlying given plane rooted tree  $t$ . Decompose  $(t, \leq)$  into a vertical composition of forests as in (1.7.1). Then the following diagram commutes

$$\begin{array}{ccccccc} s\mathcal{Q}^{\otimes n} & \xrightarrow{1^{\otimes \alpha_1} \otimes b_{k_1} \otimes 1^{\otimes \beta_1}} & s\mathcal{Q}^{\otimes \alpha_1} \otimes s\mathcal{F}_{t_{k_1}}\mathcal{Q} \otimes s\mathcal{Q}^{\otimes \beta_1} & \xrightarrow{1^{\otimes \alpha_2} \otimes b_{k_2} \otimes 1^{\otimes \beta_2}} & \dots & \xrightarrow{b_{k_N}} & s\mathcal{F}_t\mathcal{Q} \\ f_1^{\otimes n} \downarrow & & \widehat{f}_1^{\otimes \alpha_1 + 1 + \beta_1} \downarrow & & \widehat{f}_1^{\otimes \alpha_2 + 1 + \beta_2} \downarrow & \dots & \widehat{f}_1^{\otimes k_N} \downarrow \\ s\mathcal{A}^{\otimes n} & \xrightarrow{1^{\otimes \alpha_1} \otimes b_{k_1} \otimes 1^{\otimes \beta_1}} & s\mathcal{A}^{\otimes \alpha_1} \otimes s\mathcal{A} \otimes s\mathcal{A}^{\otimes \beta_1} & \xrightarrow{1^{\otimes \alpha_2} \otimes b_{k_2} \otimes 1^{\otimes \beta_2}} & \dots & \xrightarrow{b_{k_N}} & s\mathcal{A} \end{array}$$

The upper row consists of invertible maps. One can prove by induction that the composition of maps in the upper row equals  $\pm s^{-|t|}$ . When  $(t, \leq) = t_<$  is the linearly ordered tree, canonically associated with  $t$ , then the composition of maps in the upper row equals  $s^{-|t|}$ . This is also proved by induction: if  $t$  is presented as  $t = (t_1 \sqcup \dots \sqcup t_k) \cdot \mathbf{t}_k$ , then the composition of maps in the upper row is

$$\begin{aligned} (1^{\otimes k-1} \otimes s^{-|t_k|})(1^{\otimes k-2} \otimes s^{-|t_{k-1}|} \otimes 1) \dots (s^{-|t_1|} \otimes 1^{\otimes k-1})(s^{|t_1|} \otimes \dots \otimes s^{|t_{k-1}|} \otimes s^{|t_k| - |t|}) \\ = 1^{\otimes k-1} \otimes s^{-|t|} = s^{-|t|}. \end{aligned}$$

Therefore, for an arbitrary tree  $t \in \mathcal{T}_{\geq 2}^n$  the map  $\widehat{f}_1|_{s\mathcal{F}_t\mathcal{Q}}$  is

$$\widehat{f}_1 = (s\mathcal{F}_t\mathcal{Q} \xrightarrow{s^{|t|}} s\mathcal{Q}^{\otimes n} \xrightarrow{f_1^{\otimes n}} s\mathcal{A}^{\otimes n} \xrightarrow{1^{\otimes \alpha_1} \otimes b_{k_1} \otimes 1^{\otimes \beta_1}} s\mathcal{A}^{\otimes \alpha_1 + 1 + \beta_1} \xrightarrow{1^{\otimes \alpha_2} \otimes b_{k_2} \otimes 1^{\otimes \beta_2}} \dots \xrightarrow{b_{k_N}} s\mathcal{A}),$$

where the factors correspond to decomposition (1.7.1) of  $t_<$ .

**2.7. Transformations between functors from a free  $A_\infty$ -category.** Let  $\mathcal{Q}$  be a differential graded quiver, and let  $\mathcal{A}$  be an  $A_\infty$ -category. Then  $A_1(\mathcal{Q}, \mathcal{A})$  is an  $A_\infty$ -category as well. The differential graded quiver  $(sA_1(\mathcal{Q}, \mathcal{A}), B_1)$  is described as follows. Objects are chain quiver maps  $\phi : (s\mathcal{Q}, b_1) \rightarrow (s\mathcal{A}, b_1)$ , the graded  $\mathbb{k}$ -module of morphisms  $\phi \rightarrow \psi$  is the product of graded  $\mathbb{k}$ -modules

$$sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi) = \prod_{X \in \text{Ob } \mathcal{Q}} s\mathcal{A}(X\phi, X\psi) \times \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathcal{C}(s\mathcal{Q}(X, Y), s\mathcal{A}(X\phi, Y\psi)), \quad r = (r_0, r_1).$$

The differential  $B_1$  is given by

$$\begin{aligned}(rB_1)_0 &= r_0b_1, \\ (rB_1)_1 &= r_1b_1 + (\phi_1 \otimes r_0)b_2 + (r_0 \otimes \psi_1)b_2 - (-)^r b_1 r_1.\end{aligned}\tag{2.7.1}$$

Restrictions  $\phi, \psi : \mathcal{Q} \rightarrow \mathcal{A}$  of arbitrary  $A_\infty$ -functors  $\phi, \psi : \mathcal{FQ} \rightarrow \mathcal{A}$  to  $\mathcal{Q}$  are  $A_1$ -functors (chain quiver maps).

**2.8 Proposition.** *Let  $\phi, \psi : \mathcal{FQ} \rightarrow \mathcal{A}$  be  $A_\infty$ -functors. For an arbitrary complex  $P$  of  $\mathbb{k}$ -modules chain maps  $u : P \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  are in bijection with the following data:  $(u', u_k)_{k>1}$*

1. a chain map  $u' : P \rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$ ,
2.  $\mathbb{k}$ -linear maps

$$u_k : P \rightarrow \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi))$$

of degree 0 for all  $k > 1$ .

The bijection maps  $u$  to  $(u', u_k)_{k>1}$ , where  $u_k = u \cdot \text{pr}_k$  and

$$u' = (P \xrightarrow{u} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\text{restr}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\text{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)). \tag{2.8.1}$$

The inverse bijection can be recovered from the recurrent formula

$$\begin{aligned}(-)^p b_k^{\mathcal{FQ}}(pu_1) &= -(pd)u_k + \sum_{a+q+c=k}^{\alpha, \beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta}^{\mathcal{A}} \\ &\quad - (-)^p \sum_{\alpha+q+\beta=k}^{\alpha+\beta>0} (1^{\otimes \alpha} \otimes b_q^{\mathcal{FQ}} \otimes 1^{\otimes \beta})(pu_{\alpha+1+\beta}) : (s\mathcal{FQ})^{\otimes k} \rightarrow s\mathcal{A},\end{aligned}$$

where  $k > 1$ ,  $p \in P$ , and  $\phi_{a\alpha}, \psi_{c\beta}$  are matrix elements of  $\phi, \psi$ .

*Proof.* Since the  $\mathbb{k}$ -module of  $(\phi, \psi)$ -coderivations  $sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  is a product,  $\mathbb{k}$ -linear maps  $u : P \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  of degree 0 are in bijection with sequences of  $\mathbb{k}$ -linear maps  $(u_k)_{k \geq 0}$  of degree 0:

$$\begin{aligned}u_0 : P &\rightarrow \prod_{X \in \text{Ob } \mathcal{Q}} s\mathcal{A}(X\phi, X\psi), & p &\mapsto pu_0, \\ u_k : P &\rightarrow \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)), & p &\mapsto pu_k,\end{aligned}$$



for  $k \geq 1$ . The complex  $\Phi_0 = (sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi), B_1)$  admits a filtration by subcomplexes

$$\Phi_n = 0 \times \cdots \times 0 \times \prod_{k=n}^{\infty} \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)).$$

In particular,  $\Phi_2$  is a subcomplex, and

$$\Phi_0/\Phi_2 = \prod_{X \in \text{Ob } \mathcal{Q}} s\mathcal{A}(X\phi, X\psi) \times \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}(s\mathcal{FQ}(X, Y), s\mathcal{A}(X\phi, Y\psi))$$

is the quotient complex with differential (2.7.1). Since  $(s\mathcal{FQ}, b_1)$  splits into a direct sum of two subcomplexes  $s\mathcal{Q} \oplus (\oplus_{|t|>0} s\mathcal{F}_t\mathcal{Q})$ , the complex  $\Phi_0/\Phi_2$  has a subcomplex

$$(0 \times \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}(\oplus_{|t|>0} s\mathcal{F}_t\mathcal{Q}(X, Y), s\mathcal{A}(X\phi, Y\psi)), [-, b_1]).$$

The corresponding quotient complex is  $sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$ . The resulting quotient map  $\text{restr}_1 : sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$  is the restriction map. Denoting  $u' = u \cdot \text{restr}_1$ , we get the discussed assignment  $u \mapsto (u', u_n)_{n \geq 1}$ . The claim is that if  $u$  is a chain map, then the missing part

$$u_1'' : P \rightarrow \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}(\oplus_{|t|>0} s\mathcal{F}_t\mathcal{Q}(X, Y), s\mathcal{A}(X\phi, Y\psi)),$$

of  $u_1 = u_1' \times u_1''$  is recovered in a unique way.

Let us prove that the map  $u \mapsto (u', u_n)_{n \geq 1}$  is injective. The chain map  $u$  satisfies  $pdu = puB_1$  for all  $p \in P$ . That is,  $pdu_k = (puB_1)_k$  for all  $k \geq 0$ . Since  $puB_1 = (pu)b^A - (-)^p b^{\mathcal{FQ}}(pu)$ , these conditions can be rewritten as

$$pdu_k = \sum_{a+q+c=k}^{\alpha, \beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta}^A - (-)^p \sum_{\alpha+q+\beta=k} (1^{\otimes \alpha} \otimes b_q \otimes 1^{\otimes \beta})(pu_{\alpha+1+\beta}), \quad (2.8.2)$$

where  $\phi_{a\alpha} : T^a s\mathcal{FQ}(X, Y) \rightarrow T^\alpha s\mathcal{A}(X\phi, Y\phi)$  are matrix elements of  $\phi$ , and  $\psi_{c\beta}$  are matrix elements of  $\psi$ . The same formula can be rewritten as

$$\begin{aligned} (-)^p b_k^{\mathcal{FQ}}(pu_1) &= -(pd)u_k + \sum_{a+q+c=k}^{\alpha, \beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta}^A \\ &\quad - (-)^p \sum_{\alpha+q+\beta=k}^{\alpha+\beta>0} (1^{\otimes \alpha} \otimes b_q^{\mathcal{FQ}} \otimes 1^{\otimes \beta})(pu_{\alpha+1+\beta}) : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \rightarrow s\mathcal{A}. \end{aligned} \quad (2.8.3)$$

When  $k > 1$ , the map  $b_k^{\mathcal{FQ}} : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \cdots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \rightarrow s\mathcal{F}_t\mathcal{Q}$ ,  $t = (t_1 \sqcup \cdots \sqcup t_k)\mathbf{t}_k$  is invertible, thus,  $pu_1 : s\mathcal{F}_t\mathcal{Q} \rightarrow s\mathcal{A}$  in the left hand side is determined in a unique way by  $u_0, u_n$  for

$n > 1$  and by  $pu_1 : s\mathcal{F}_{t_i}\mathcal{Q} \rightarrow s\mathcal{A}$ ,  $1 \leq i \leq k$ , occurring in the right hand side. Since the restriction  $u'_1$  of  $u_1$  to  $s\mathcal{F}_1\mathcal{Q} = s\mathcal{Q}$  is known by 1), the map  $u'_1$  is recursively recovered from  $(u_0, u'_1, u_n)_{n>1}$ .

Let us prove that the map  $u \mapsto (u', u_n)_{n>1}$  is surjective. Given  $(u_0, u'_1, u_n)_{n>1}$  we define maps  $u''_1$  of degree 0 recursively by (2.8.3). This implies equation (2.8.2) for  $k > 1$ . For  $k = 0$  this equation in the form  $pdu_0 = pu_0b_1$  holds due to condition 1). It remains to prove equation (2.8.2) for  $k = 1$ :

$$(pd)u_1 = (pu_1)b_1^A + (\phi_1 \otimes pu_0)b_2^A + (pu_0 \otimes \psi_1)b_2^A - (-)^pb_1(pu_1) : s\mathcal{F}_t\mathcal{Q}(X, Y) \rightarrow s\mathcal{A}(X\phi, Y\psi) \quad (2.8.4)$$

for all trees  $t \in \mathcal{T}_{\geq 2}$ . For  $t = |$  it holds due to assumption 1). Let  $N > 1$  be an integer. Assume that equation (2.8.4) holds for all trees  $t \in \mathcal{T}_{\geq 2}$  with the number of input leaves  $\text{in}(t) < N$ . Let  $t \in \mathcal{T}_{\geq 2}^N$  be a tree (with  $\text{in}(t) = N$ ). Then  $t = (t_1 \sqcup \dots \sqcup t_k)\mathbf{t}_k$  for some  $k > 1$  and some trees  $t_i \in \mathcal{T}_{\geq 2}$ ,  $\text{in}(t_i) < N$ . For such  $t$  equation (2.8.4) is equivalent to

$$\begin{aligned} (-)^pb_k(pd)u_1 &= (-)^pb_k(pu_1)b_1^A + (-)^pb_k(\phi_1 \otimes pu_0)b_2^A + (-)^pb_k(pu_0 \otimes \psi_1)b_2^A \\ &+ \sum_{\substack{\gamma+\delta>0 \\ \gamma+j+\delta=k}} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})b_{\gamma+1+\delta}(pu_1) : s\mathcal{F}_{t_1}\mathcal{Q} \otimes \dots \otimes s\mathcal{F}_{t_k}\mathcal{Q} \rightarrow s\mathcal{A}. \end{aligned}$$

Substituting definition (2.8.3) of  $u_1$  we turn the above equation into an identity

$$- \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pdu_q \otimes \psi_{c\beta})b_{\alpha+1+\beta}^A \quad (2.8.5)$$

$$- (-)^p \sum_{\substack{\alpha+\beta>0 \\ \alpha+q+\beta=k}} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pdu_{\alpha+1+\beta}) \quad (2.8.6)$$

$$= -(pdu_k)b_1 \quad (2.8.7)$$

$$\begin{aligned} &+ \sum_{a+q+c=k}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta})b_{\alpha+1+\beta}b_1 \\ &- (-)^p \sum_{\substack{\alpha+\beta>0 \\ \alpha+q+\beta=k}} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta})b_1 \end{aligned} \quad (2.8.8)$$

$$+ (-)^pb_k(\phi_1 \otimes pu_0)b_2 + (-)^pb_k(pu_0 \otimes \psi_1)b_2 \quad (2.8.9)$$

$$+ (-)^p \sum_{\substack{\gamma+\delta>0 \\ \gamma+j+\delta=k}} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta}) \left[ -pdu_{\gamma+1+\delta} \right] \quad (2.8.10)$$

$$+ \sum_{a+q+c=\gamma+1+\delta}^{\alpha,\beta} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta})b_{\alpha+1+\beta} \quad (2.8.11)$$

$$- (-)^p \sum_{\alpha+q+\beta=\gamma+1+\delta}^{\alpha+\beta>0} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}) \Big], \quad (2.8.12)$$

whose validity we are going to prove now. First of all, terms (2.8.6) and (2.8.10) cancel each other. Term (2.8.12) vanishes because for an arbitrary integer  $g$  the sum

$$\sum_{\substack{\alpha+1+\beta=g \\ \gamma+j+\delta=k \\ \alpha+q+\beta=\gamma+1+\delta}} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta}) \quad (2.8.13)$$

is the matrix coefficient  $b^2 = 0 : T^k s\mathcal{FQ} \rightarrow T^g s\mathcal{FQ}$ , thus, it vanishes. Notice that condition  $\alpha + \beta > 0$  in (2.8.12) automatically implies  $\gamma + \delta > 0$ . Furthermore, term (2.8.7) cancels one of the terms of sum (2.8.5). In the remaining terms of (2.8.5) we may use the induction assumptions and replace  $pdu_q$  with the right hand side of (2.8.2). We also absorb terms (2.8.9) into sum (2.8.11), allowing  $\gamma = \delta = 0$  in it and allowing simultaneously  $\alpha = \beta = 0$  in (2.8.8) to compensate for the missing term  $b_k(pu_1)b_1$ :

$$- \sum_{\alpha+\beta>0} \sum_{\substack{\gamma,\delta \\ a+q+c=k \\ e+j+f=q}} [\phi_{a\alpha} \otimes (\phi_{e\gamma} \otimes pu_j \otimes \psi_{f\delta}) b_{\gamma+1+\delta}^A \otimes \psi_{c\beta}] b_{\alpha+1+\beta}^A \quad (2.8.14)$$

$$+ (-)^p \sum_{\alpha+\beta>0} \sum_{\substack{\gamma,\delta \\ a+q+c=k \\ \gamma+j+\delta=q}} [\phi_{a\alpha} \otimes (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(pu_{\gamma+1+\delta}) \otimes \psi_{c\beta}] b_{\alpha+1+\beta}^A \quad (2.8.15)$$

$$= \sum_{\substack{\alpha,\beta \\ a+q+c=k}} (\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta}^A b_1^A \quad (2.8.16)$$

$$- (-)^p \sum_{\alpha+q+\beta=k} (1^{\otimes\alpha} \otimes b_q \otimes 1^{\otimes\beta})(pu_{\alpha+1+\beta}) b_1^A \quad (2.8.17)$$

$$+ (-)^p \sum_{\gamma+j+\delta=k} \sum_{\substack{\alpha,\beta \\ a+q+c=\gamma+1+\delta}} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(\phi_{a\alpha} \otimes pu_q \otimes \psi_{c\beta}) b_{\alpha+1+\beta}^A.$$

Recall that  $\phi_{a0}$  vanish for all  $a$  except  $a = 0$ . Therefore, we may absorb term (2.8.16) into sum (2.8.14) and term (2.8.17) into sum (2.8.15), allowing terms with  $\alpha = \beta = 0$  in them. Denote  $r = pu \in sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$ . The proposition follows immediately from the following

**2.9 Lemma.** *For all  $r \in sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  and all  $k \geq 0$  we have*

$$\begin{aligned} & - \sum_{\alpha,\beta} \sum_{\substack{\gamma,\delta \\ a+q+c=k \\ e+j+f=q}} [\phi_{a\alpha} \otimes (\phi_{e\gamma} \otimes r_j \otimes \psi_{f\delta}) b_{\gamma+1+\delta}^A \otimes \psi_{c\beta}] b_{\alpha+1+\beta}^A \\ & + (-)^r \sum_{\alpha,\beta} \sum_{\substack{\gamma,\delta \\ a+q+c=k \\ \gamma+j+\delta=q}} [\phi_{a\alpha} \otimes (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta}) r_{\gamma+1+\delta} \otimes \psi_{c\beta}] b_{\alpha+1+\beta}^A \end{aligned}$$

$$= (-)^r \sum_{\gamma+j+\delta=k} \sum_{a+q+c=\gamma+1+\delta}^{\alpha,\beta} (1^{\otimes\gamma} \otimes b_j \otimes 1^{\otimes\delta})(\phi_{a\alpha} \otimes r_q \otimes \psi_{c\beta})b_{\alpha+1+\beta}^A. \quad (2.9.1)$$

*Proof.* Sum (2.9.1) is split into three sums accordingly to output of  $b_j$  being an input of  $\phi_{a\alpha}$  or  $r_q$  or  $\psi_{c\beta}$ :

$$- \sum_{a+e+j+f+c=k}^{\alpha,\beta,\gamma,\delta} (\phi_{a\alpha} \otimes \phi_{e\gamma} \otimes r_j \otimes \psi_{f\delta} \otimes \psi_{c\beta})(1^{\otimes\alpha} \otimes b_{\gamma+1+\delta}^A \otimes 1^{\otimes\beta})b_{\alpha+1+\beta}^A \quad (2.9.2)$$

$$+ (-)^r \sum_{a+\gamma+j+\delta+c=k}^{\alpha,\beta} (1^{\otimes a+\gamma} \otimes b_j \otimes 1^{\otimes\delta+c})(\phi_{a\alpha} \otimes r_{\gamma+1+\delta} \otimes \psi_{c\beta})b_{\alpha+1+\beta}^A \quad (2.9.3)$$

$$= (-)^r \sum_{x+q+c=k}^{a,\alpha,\beta} (b_{xa}\phi_{a\alpha} \otimes r_q \otimes \psi_{c\beta})b_{\alpha+1+\beta}^A \quad (2.9.4)$$

$$+ (-)^r \sum_{a+y+c=k}^{\alpha,q,\beta} (\phi_{a\alpha} \otimes b_{yq}r_q \otimes \psi_{c\beta})b_{\alpha+1+\beta}^A \quad (2.9.5)$$

$$+ \sum_{a+q+z=k}^{\alpha,\beta,c} (\phi_{a\alpha} \otimes r_q \otimes b_{zc}\psi_{c\beta})b_{\alpha+1+\beta}^A. \quad (2.9.6)$$

Here  $b_{xa} : T^x s\mathcal{FQ} \rightarrow T^a s\mathcal{FQ}$  is a matrix element of  $b^{\mathcal{FQ}}$ . Terms (2.9.3) and (2.9.5) cancel each other. We shall use  $A_\infty$ -functor identities  $b\phi = \phi b$ ,  $b\psi = \psi b$  for terms (2.9.4) and (2.9.6). Being a cocategory homomorphism,  $\phi$  satisfies the identity

$$\sum_{a+e=h} \phi_{a\alpha} \otimes \phi_{e\gamma} = [\Delta(\phi \otimes \phi)]_{h;\alpha,\gamma} = \phi_{h,\alpha+\gamma} \Delta_{\alpha+\gamma;\alpha,\gamma}$$

for all non-negative integers  $h$ , where  $\Delta$  is the cut comultiplication. Similarly for  $\psi$ . Using this identity in (2.9.2) we get the equation to verify:

$$\begin{aligned} & - \sum_{x+q+z=k}^{v,w} (\phi_{xv} \otimes r_q \otimes \psi_{zw}) \sum_{\alpha+y+\beta=v+1+w}^{\alpha \leq v, \beta \leq w} (1^{\otimes\alpha} \otimes b_y^A \otimes 1^{\otimes\beta})b_{\alpha+1+\beta}^A \\ & = \sum_{x+q+z=k}^{v,w,\alpha} (\phi_{xv} \otimes r_q \otimes \psi_{zw})(b_{v\alpha}^A \otimes 1^{\otimes 1+w})b_{\alpha+1+w}^A \\ & + \sum_{x+q+z=k}^{v,w,\beta} (\phi_{xv} \otimes r_q \otimes \psi_{zw})(1^{\otimes v+1} \otimes b_{w\beta}^A)b_{v+1+\beta}^A. \end{aligned}$$

It follows from the identity  $b^2 \text{pr}_1 = 0 : T^{v+1+w} s\mathcal{A} \rightarrow s\mathcal{A}$  valid for arbitrary non-negative integers  $v, w$ , which we may rewrite like this:

$$\sum_{\alpha+y+\beta=v+1+w}^{\alpha \leq v, \beta \leq w} (1^{\otimes\alpha} \otimes b_y^A \otimes 1^{\otimes\beta})b_{\alpha+1+\beta}^A + \sum_{\alpha} (b_{v\alpha}^A \otimes 1^{\otimes 1+w})b_{\alpha+1+w}^A + \sum_{\beta} (1^{\otimes v+1} \otimes b_{w\beta}^A)b_{v+1+\beta}^A = 0.$$

So the lemma is proved.  $\square$

The proposition follows.  $\square$

Let us consider now the question, when the discussed chain map is null-homotopic.

**2.10 Corollary.** *Let  $\phi, \psi : \mathcal{FQ} \rightarrow \mathcal{A}$  be  $A_\infty$ -functors. Let  $P$  be a complex of  $\mathbb{k}$ -modules. Let  $u : P \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  be a chain map. The set (possibly empty) of homotopies  $h : P \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$ ,  $\deg h = -1$ , such that  $u = dh + hB_1$  is in bijection with the set of data  $(h', h_k)_{k>1}$ , consisting of*

1. a homotopy  $h' : P \rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)$ ,  $\deg h' = -1$ , such that  $dh' + h'B_1 = u'$ , where  $u'$  is given by (2.8.1);
2.  $\mathbb{k}$ -linear maps

$$h_k : P \rightarrow \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathbb{C}((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi))$$

of degree  $-1$  for all  $k > 1$ .

The bijection maps  $h$  to  $(h', h_k)_{k>1}$ , where  $h_k = h \cdot \text{pr}_k$  and

$$h' = (P \xrightarrow{h} sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\text{restr}} sA_1(\mathcal{FQ}, \mathcal{A})(\phi, \psi) \xrightarrow{\text{restr}} sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi)).$$

The inverse bijection can be recovered from the recurrent formula

$$\begin{aligned} (-)^p b_k(ph_1) &= pu_k - (pd)h_k - \sum_{a+q+c=k}^{\alpha, \beta} (\phi_{a\alpha} \otimes ph_q \otimes \psi_{c\beta})b_{\alpha+1+\beta} \\ &\quad - (-)^p \sum_{a+q+c=k}^{a+c>0} (1^{\otimes a} \otimes b_q \otimes 1^{\otimes c})(ph_{a+1+c}) : (s\mathcal{FQ})^{\otimes k} \rightarrow s\mathcal{A}, \end{aligned}$$

where  $k > 1$ ,  $p \in P$ , and  $\phi_{a\alpha}, \psi_{c\beta}$  are matrix elements of  $\phi, \psi$ .

*Proof.* We shall apply Proposition 2.8 to the complex  $\text{Cone}(\text{id} : P \rightarrow P)$  instead of  $P$ . The graded  $\mathbb{k}$ -module  $\text{Cone}(\text{id}_P) = P \oplus P[1]$  is equipped with the differential  $(q, ps)d = (qd + p, -pds)$ ,  $p, q \in P$ . The chain maps  $\bar{u} : \text{Cone}(\text{id}_P) \rightarrow C$  to an arbitrary complex  $C$  are in bijection with pairs  $(u : P \rightarrow C, h : P \rightarrow C)$ , where  $u = dh + hd$  and  $\deg h = -1$ . The pair  $(u, h) = (\text{in}_1 \bar{u}, \text{sin}_2 \bar{u})$  is assigned to  $\bar{u}$ , and the map  $\bar{u} : P \oplus P[1] \rightarrow C$ ,  $(q, ps) \mapsto qu + ph$  is assigned to a pair  $(u, h)$ . Indeed,  $\bar{u}$  being chain map is equivalent to

$$(q, ps)d\bar{u} = qdu + pu - pdh = qud + phd = (q, ps)\bar{u}d,$$

that is, to conditions  $du = ud$ ,  $u = dh + hd$ .

Thus, for a fixed chain map  $u : P \rightarrow C$  the set of homotopies  $h : P \rightarrow C$ , such that  $u = dh + hd$ , is in bijection with the set of chain maps  $\bar{u} : \text{Cone}(\text{id}_P) \rightarrow C$  such that  $\text{in}_1 \bar{u} = u : P \rightarrow C$ . Applying this statement to  $u : P \rightarrow C = sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  we find by Proposition 2.8 that the set of homotopies  $h : P \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(\phi, \psi)$  such that  $u = dh + hB_1$  is in bijection with the set of data  $(\bar{u}', \bar{u}_k)_{k>1}$ , such that

$$\begin{aligned} \bar{u}' : \text{Cone}(\text{id}_P) &\rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\phi, \psi) && \text{is a chain map,} && \text{in}_1 \bar{u}' = u', \\ \bar{u}_k : \text{Cone}(\text{id}_P) &\rightarrow \prod_{X, Y \in \text{Ob } \mathcal{Q}} \mathcal{C}((s\mathcal{FQ})^{\otimes k}(X, Y), s\mathcal{A}(X\phi, Y\psi)), && \deg \bar{u}_k = 0, && \text{in}_1 \bar{u}_k = u_k, \end{aligned}$$

therefore, in bijection with the set of data  $(h', h_k)_{k>1} = (s \text{in}_2 \bar{u}', s \text{in}_2 \bar{u}_k)_{k>1}$ , as stated in corollary.  $\square$

**2.11. Restriction as an  $A_\infty$ -functor.** Let  $\mathcal{Q}$  be a ( $\mathcal{U}$ -small) differential graded  $\mathbb{k}$ -quiver. Denote by  $\mathcal{FQ}$  the free  $A_\infty$ -category generated by  $\mathcal{Q}$ . Let  $\mathcal{A}$  be a ( $\mathcal{U}$ -small) unital  $A_\infty$ -category. There is the restriction strict  $A_\infty$ -functor

$$\text{restr} : A_\infty(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A}), \quad (f : \mathcal{FQ} \rightarrow \mathcal{A}) \mapsto (\bar{f} = (f_1|_{\mathcal{Q}}) : \mathcal{Q} \rightarrow \mathcal{A}).$$

In fact, it is the composition of two strict  $A_\infty$ -functors:  $A_\infty(\mathcal{FQ}, \mathcal{A}) \xrightarrow{\text{restr}_{\infty, 1}} A_1(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A})$ , where the second comes from the full embedding  $\mathcal{Q} \hookrightarrow \mathcal{FQ}$ . Its first component is

$$\begin{aligned} \text{restr}_1 : sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g) &\rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g}), \\ r = (r_0, r_1, \dots, r_n, \dots) &\mapsto (r_0, r_1|_{\mathcal{Q}}) = \bar{r}. \end{aligned} \tag{2.11.1}$$

**2.12 Theorem.** *The  $A_\infty$ -functor  $\text{restr} : A_\infty(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A})$  is an equivalence.*

*Proof.* Let us prove that restriction map (2.11.1) is homotopy invertible. We construct a chain map going in the opposite direction

$$u : sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g}) \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g)$$

via Proposition 2.8 taking  $P = sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g})$ . We choose

$$u' : sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g}) \rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g})$$

to be the identity map and  $u_k = 0$  for  $k > 1$ . Therefore,

$$u \cdot \text{restr}_1 = u' = \text{id}_{sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g})}.$$

Denote

$$v = \text{id}_{sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g)} - [sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g) \xrightarrow{\text{restr}_1} sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g}) \xrightarrow{u} sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g)].$$

Let us prove that  $v$  is null-homotopic via Corollary 2.10, taking  $P = sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g)$ . A homotopy  $h : sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g) \rightarrow sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g)$ ,  $\deg h = -1$ , such that  $v = B_1h + hB_1$  is specified by  $h' = 0 : sA_\infty(\mathcal{FQ}, \mathcal{A})(f, g) \rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\bar{f}, \bar{g})$  and  $h_k = 0$  for  $k > 1$ . Indeed,

$$v' = v \cdot \text{restr}_1 = \text{restr}_1 - \text{restr}_1 \cdot u \cdot \text{restr}_1 = \text{restr}_1 - \text{restr}_1 = 0,$$

so  $v' = B_1h' + h'B_1$  and condition 1 of Corollary 2.10 is satisfied<sup>2</sup>. Therefore,  $u$  is homotopy inverse to  $\text{restr}_1$ .

Let  $\mathbf{i}^{\mathcal{A}}$  be a unit transformation of the unital  $A_\infty$ -category  $\mathcal{A}$ . Then  $A_1(\mathcal{Q}, \mathcal{A})$  is a unital  $A_\infty$ -category with the unit transformation  $(1 \otimes \mathbf{i}^{\mathcal{A}})M$  (cf. [Lyu03, Proposition 7.7]). The unit element for an object  $\phi \in \text{Ob } A_1(\mathcal{Q}, \mathcal{A})$  is  ${}_{\phi}\mathbf{i}_0^{A_1(\mathcal{Q}, \mathcal{A})} : \mathbb{k} \rightarrow sA_1(\mathcal{Q}, \mathcal{A})$ ,  $1 \mapsto \phi\mathbf{i}^{\mathcal{A}}$ . The  $A_\infty$ -category  $A_\infty(\mathcal{FQ}, \mathcal{A})$  is also unital. To establish equivalence of these two  $A_\infty$ -categories via  $\text{restr} : A_\infty(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A})$  we verify the conditions of Theorem 8.8 from [Lyu03].

Consider the mapping  $\text{Ob } A_1(\mathcal{Q}, \mathcal{A}) \rightarrow \text{Ob } A_\infty(\mathcal{FQ}, \mathcal{A})$ ,  $\phi \mapsto \widehat{\phi}$ , which extends a given chain map to a strict  $A_\infty$ -functor, constructed in Corollary 2.4. Clearly,  $\overline{\widehat{\phi}} = \phi$ . It remains to give two mutually inverse cycles, which we choose as follows:

$$\begin{aligned} {}_{\phi}r_0 : \mathbb{k} &\rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\phi, \widehat{\widehat{\phi}}), & 1 &\mapsto \phi\mathbf{i}^{\mathcal{A}}, \\ {}_{\phi}p_0 : \mathbb{k} &\rightarrow sA_1(\mathcal{Q}, \mathcal{A})(\widehat{\widehat{\phi}}, \phi), & 1 &\mapsto \phi\mathbf{i}^{\mathcal{A}}. \end{aligned}$$

Clearly,  ${}_{\phi}r_0B_1 = 0$ ,  ${}_{\phi}p_0B_1 = 0$ ,

$$\begin{aligned} ({}_{\phi}r_0 \otimes {}_{\phi}p_0)B_2 - {}_{\phi}\mathbf{i}_0^{A_1(\mathcal{Q}, \mathcal{A})} : 1 &\mapsto (\phi\mathbf{i}^{\mathcal{A}} \otimes \phi\mathbf{i}^{\mathcal{A}})B_2 - \phi\mathbf{i}^{\mathcal{A}} \in \text{Im } B_1, \\ ({}_{\phi}p_0 \otimes {}_{\phi}r_0)B_2 - {}_{\phi}\mathbf{i}_0^{A_1(\mathcal{Q}, \mathcal{A})} : 1 &\mapsto (\phi\mathbf{i}^{\mathcal{A}} \otimes \phi\mathbf{i}^{\mathcal{A}})B_2 - \phi\mathbf{i}^{\mathcal{A}} \in \text{Im } B_1. \end{aligned}$$

Therefore, all assumptions of Theorem 8.8 [Lyu03] are satisfied. Thus,  $\text{restr} : A_\infty(\mathcal{FQ}, \mathcal{A}) \rightarrow A_1(\mathcal{Q}, \mathcal{A})$  is an  $A_\infty$ -equivalence.  $\square$

**2.13 Corollary.** *Every  $A_\infty$ -functor  $f : \mathcal{FQ} \rightarrow \mathcal{A}$  is isomorphic to the strict  $A_\infty$ -functor  $\widehat{\widehat{f}} : \mathcal{FQ} \rightarrow \mathcal{A}$ .*

*Proof.* Note that  $\bar{f} = \widehat{\widehat{f}}$ . The  $A_1$ -transformation  $\bar{f}\mathbf{i}^{\mathcal{A}} : \bar{f} \rightarrow \widehat{\widehat{f}} : \mathcal{Q} \rightarrow \mathcal{A}$  with the components  $({}_xf\mathbf{i}_0^{\mathcal{A}}, \bar{f}_1\mathbf{i}_1^{\mathcal{A}})$  is natural. It is mapped by  $u$  into a natural  $A_\infty$ -transformation  $(\bar{f}\mathbf{i}^{\mathcal{A}})u : f \rightarrow \widehat{\widehat{f}} : \mathcal{FQ} \rightarrow \mathcal{A}$ . Its zero component  ${}_xf\mathbf{i}_0^{\mathcal{A}}$  is invertible, therefore  $(\bar{f}\mathbf{i}^{\mathcal{A}})u$  is invertible by [Lyu03, Proposition 7.15].  $\square$

---

<sup>2</sup>By the way, the only non-vanishing component of  $h$  is  $h_1$ .

### 3. Representable 2-functors $A_\infty^u \rightarrow A_\infty^u$

Recall that unital  $A_\infty$ -categories, unital  $A_\infty$ -functors and equivalence classes of natural  $A_\infty$ -transformations form a 2-category [Lyu03]. In order to distinguish between the  $A_\infty$ -category  $A_\infty^u(\mathcal{C}, \mathcal{D})$  and the ordinary category, whose morphisms are equivalence classes of natural  $A_\infty$ -transformations, we denote the latter by

$$\overline{A_\infty^u}(\mathcal{C}, \mathcal{D}) = H^0(A_\infty^u(\mathcal{C}, \mathcal{D}), m_1).$$

The corresponding notation for the 2-category is  $\overline{A_\infty^u}$ . We will see that arbitrary  $A_N$ -categories can be viewed as 2-functors  $\overline{A_\infty^u} \rightarrow \overline{A_\infty^u}$ . Moreover, they come from certain generalizations called  $A_\infty^u$ -2-functors. There is a notion of representability of such 2-functors, which explains some constructions of  $A_\infty$ -categories. For instance, a differential graded  $\mathbb{k}$ -quiver  $\mathcal{Q}$  will be represented by the free  $A_\infty$ -category  $\mathcal{FQ}$  generated by it.

**3.1 Definition.** A (strict)  $A_\infty^u$ -2-functor  $F : A_\infty^u \rightarrow A_\infty^u$  consists of

1. a map  $F : \text{Ob } A_\infty^u \rightarrow \text{Ob } A_\infty^u$ ;
2. a unital  $A_\infty$ -functor  $F = F_{\mathcal{C}, \mathcal{D}} : A_\infty^u(\mathcal{C}, \mathcal{D}) \rightarrow A_\infty^u(F\mathcal{C}, F\mathcal{D})$  for each pair  $\mathcal{C}, \mathcal{D}$  of unital  $A_\infty$ -categories;  
such that
3.  $\text{id}_{F\mathcal{C}} = F(\text{id}_{\mathcal{C}})$  for any unital  $A_\infty$ -category  $\mathcal{C}$ ;
4. the equation

$$\begin{array}{ccc} TsA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty^u(\mathcal{D}, \mathcal{E}) & \xrightarrow{M} & TsA_\infty^u(\mathcal{C}, \mathcal{E}) \\ \downarrow F_{\mathcal{C}, \mathcal{D}} \boxtimes F_{\mathcal{D}, \mathcal{E}} & = & \downarrow F_{\mathcal{C}, \mathcal{E}} \\ TsA_\infty^u(F\mathcal{C}, F\mathcal{D}) \boxtimes TsA_\infty^u(F\mathcal{D}, F\mathcal{E}) & \xrightarrow{M} & TsA_\infty^u(F\mathcal{C}, F\mathcal{E}) \end{array} \quad (3.1.1)$$

holds strictly for each triple  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  of unital  $A_\infty$ -categories.

The  $A_\infty$ -functor  $F : A_\infty^u(\mathcal{C}, \mathcal{D}) \rightarrow A_\infty^u(F\mathcal{C}, F\mathcal{D})$  consists of the mapping of objects

$$\text{Ob } F : \text{Ob } A_\infty^u(\mathcal{C}, \mathcal{D}) \rightarrow \text{Ob } A_\infty^u(F\mathcal{C}, F\mathcal{D}), \quad f \mapsto Ff,$$

and the components  $F_k, k \geq 1$ :

$$\begin{aligned} F_1 &: sA_\infty^u(\mathcal{C}, \mathcal{D})(f, g) \rightarrow sA_\infty^u(F\mathcal{C}, F\mathcal{D})(Ff, Fg), \\ F_2 &: sA_\infty^u(\mathcal{C}, \mathcal{D})(f, g) \otimes sA_\infty^u(\mathcal{C}, \mathcal{D})(g, h) \rightarrow sA_\infty^u(F\mathcal{C}, F\mathcal{D})(Ff, Fh), \end{aligned}$$

and so on.

Weak versions of  $A_\infty^u$ -2-functors and 2-transformations between them might be considered elsewhere.



**3.2 Definition.** A (strict)  $A_\infty^u$ -2-transformation  $\lambda : F \rightarrow G : A_\infty^u \rightarrow A_\infty^u$  of strict  $A_\infty^u$ -2-functors is

1. a family of unital  $A_\infty$ -functors  $\lambda_{\mathcal{C}} : F\mathcal{C} \rightarrow G\mathcal{C}$ ,  $\mathcal{C} \in \text{Ob } A_\infty^u$ ;  
such that
2. the diagram of  $A_\infty$ -functors

$$\begin{array}{ccc} A_\infty^u(\mathcal{C}, \mathcal{D}) & \xrightarrow{F} & A_\infty^u(F\mathcal{C}, F\mathcal{D}) \\ G \downarrow & = & \downarrow (1 \boxtimes \lambda_{\mathcal{D}})M \\ A_\infty^u(G\mathcal{C}, G\mathcal{D}) & \xrightarrow{(\lambda_{\mathcal{C}} \boxtimes 1)M} & A_\infty^u(F\mathcal{C}, G\mathcal{D}) \end{array} \quad (3.2.1)$$

strictly commutes.

An  $A_\infty^u$ -2-transformation  $\lambda = (\lambda_{\mathcal{C}})$  for which  $\lambda_{\mathcal{C}}$  are  $A_\infty$ -equivalences is called a *natural  $A_\infty^u$ -2-equivalence*.

Let us show now that the above notions induce ordinary strict 2-functors and strict 2-transformations in 0-th cohomology. Recall that the strict 2-category  $\overline{A_\infty^u}$  consists of objects – unital  $A_\infty$ -categories, the category  $\overline{A_\infty^u}(\mathcal{C}, \mathcal{D})$  for any pair of objects  $\mathcal{C}, \mathcal{D}$ , the identity functor  $\text{id}_{\mathcal{C}}$  for any unital  $A_\infty$ -category  $\mathcal{C}$ , and the composition functor [Lyu03]

$$\begin{aligned} \overline{A_\infty^u}(\mathcal{C}, \mathcal{D})(f, g) \times \overline{A_\infty^u}(\mathcal{D}, \mathcal{E})(h, k) &\xrightarrow{\bullet^2} \overline{A_\infty^u}(\mathcal{C}, \mathcal{E})(fh, gk), \\ (rs^{-1}, ps^{-1}) &\longmapsto (rhs^{-1} \otimes gps^{-1})m_2. \end{aligned}$$

Given a strict  $A_\infty^u$ -2-functor  $F$  as in Definition 3.1 we construct from it an ordinary strict 2-functor  $\overline{F} = F : \text{Ob } \overline{A_\infty^u} \rightarrow \text{Ob } \overline{A_\infty^u}$ ,  $\overline{F} = H^0(sF_1s^{-1}) : \overline{A_\infty^u}(\mathcal{C}, \mathcal{D}) \rightarrow \overline{A_\infty^u}(F\mathcal{C}, F\mathcal{D})$  as follows.

Denote

$$\begin{aligned} M_{10} \odot M_{01} &= \{ sA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes sA_\infty^u(\mathcal{D}, \mathcal{E}) \xrightarrow[\sim]{\Delta_{10} \boxtimes \Delta_{01}} \\ &\quad [sA_\infty^u(\mathcal{C}, \mathcal{D}) \otimes T^0 sA_\infty^u(\mathcal{C}, \mathcal{D})] \boxtimes [T^0 sA_\infty^u(\mathcal{D}, \mathcal{E}) \otimes sA_\infty^u(\mathcal{D}, \mathcal{E})] \\ &\xrightarrow{\sim} [sA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes T^0 sA_\infty^u(\mathcal{D}, \mathcal{E})] \otimes [T^0 sA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes sA_\infty^u(\mathcal{D}, \mathcal{E})] \\ &\xrightarrow{M_{10} \otimes M_{01}} sA_\infty^u(\mathcal{C}, \mathcal{E}) \otimes sA_\infty^u(\mathcal{C}, \mathcal{E}) \}, \end{aligned} \quad (3.2.2)$$

where the obvious isomorphisms  $\Delta_{10}$  and  $\Delta_{01}$  are components of the comultiplication  $\Delta$ , the middle isomorphism is that of distributivity law (1.5.3), and the components  $M_{10}$  and  $M_{01}$  of  $M$  are the composition maps.

Property (3.1.1) of  $F$  implies that

$$(M_{10} \odot M_{01})(F_1 \otimes F_1) = (F_1 \boxtimes F_1)(M_{10} \odot M_{01}). \quad (3.2.3)$$

Indeed, the following diagram commutes

$$\begin{array}{ccc} sA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes T^0 sA_\infty^u(\mathcal{D}, \mathcal{E}) & \xrightarrow{M_{10}} & sA_\infty^u(\mathcal{C}, \mathcal{E}) \\ F_1 \boxtimes \text{Ob } F \downarrow & & \downarrow F_1 \\ sA_\infty^u(F\mathcal{C}, F\mathcal{D}) \boxtimes T^0 sA_\infty^u(F\mathcal{D}, F\mathcal{E}) & \xrightarrow{M_{10}} & sA_\infty^u(F\mathcal{C}, F\mathcal{E}) \end{array}$$

due to (3.1.1).  $\otimes$ -tensoring it with one more similar diagram we get

$$(M_{10} \otimes M_{01})(F_1 \otimes F_1) = [(F_1 \boxtimes \text{Ob } F) \otimes (\text{Ob } F \boxtimes F_1)](M_{10} \otimes M_{01}).$$

The isomorphisms in (3.2.2) commute with  $F$  in expected way, so (3.2.3) follows.

We claim that the diagram

$$\begin{array}{ccc} sA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes sA_\infty^u(\mathcal{D}, \mathcal{E}) & \xrightarrow{(M_{10} \odot M_{01})B_2} & sA_\infty^u(\mathcal{C}, \mathcal{E}) \\ F_1 \boxtimes F_1 \downarrow & & \downarrow F_1 \\ sA_\infty^u(F\mathcal{C}, F\mathcal{D}) \boxtimes sA_\infty^u(F\mathcal{D}, F\mathcal{E}) & \xrightarrow{(M_{10} \odot M_{01})B_2} & sA_\infty^u(F\mathcal{C}, F\mathcal{E}) \end{array} \quad (3.2.4)$$

homotopically commutes. Indeed, since

$$(1 \otimes B_1 + B_1 \otimes 1)F_2 + B_2F_1 = (F_1 \otimes F_1)B_2 + F_2B_1,$$

we get

$$\begin{aligned} & (M_{10} \odot M_{01})B_2F_1 \\ &= (M_{10} \odot M_{01})(F_1 \otimes F_1)B_2 + (M_{10} \odot M_{01})F_2B_1 - (M_{10} \odot M_{01})(1 \otimes B_1 + B_1 \otimes 1)F_2 \\ &= (F_1 \boxtimes F_1)(M_{10} \odot M_{01})B_2 + (M_{10} \odot M_{01})F_2B_1 - (1 \boxtimes B_1 + B_1 \boxtimes 1)(M_{10} \odot M_{01})F_2. \end{aligned}$$

We have used equations

$$\begin{aligned} (M_{10} \odot M_{01})(1 \otimes B_1) &= (1 \boxtimes B_1)(M_{10} \odot M_{01}), \\ (M_{10} \odot M_{01})(B_1 \otimes 1) &= (B_1 \boxtimes 1)(M_{10} \odot M_{01}), \end{aligned}$$

which can be proved similarly to (3.2.3) due to  $M$  being an  $A_\infty$ -functor. Passing to cohomology we get from (3.2.4) a strictly commutative diagram of functors

$$\begin{array}{ccc} H^0(A_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes A_\infty^u(\mathcal{D}, \mathcal{E})) & \xrightarrow{\bullet^2} & H^0(A_\infty^u(\mathcal{C}, \mathcal{E})) \\ H^0(sF_1s^{-1} \boxtimes sF_1s^{-1}) \downarrow & = & \downarrow H^0(sF_1s^{-1}) \\ H^0(A_\infty^u(F\mathcal{C}, F\mathcal{D}) \boxtimes A_\infty^u(F\mathcal{D}, F\mathcal{E})) & \xrightarrow{\bullet^2} & H^0(A_\infty^u(F\mathcal{C}, F\mathcal{E})) \end{array}$$

since  $\bullet^2 = H^0((s \boxtimes s)(M_{10} \odot M_{01})B_2s^{-1})$ . Using the Künneth map we come to strictly commutative diagram of functors

$$\begin{array}{ccc} \overline{A}_\infty^u(\mathcal{C}, \mathcal{D}) \times \overline{A}_\infty^u(\mathcal{D}, \mathcal{E}) & \xrightarrow{\bullet^2} & \overline{A}_\infty^u(\mathcal{C}, \mathcal{E}) \\ H^0(sF_1s^{-1}) \times H^0(sF_1s^{-1}) \downarrow & = & \downarrow H^0(sF_1s^{-1}) \\ \overline{A}_\infty^u(F\mathcal{C}, F\mathcal{D}) \times \overline{A}_\infty^u(F\mathcal{D}, F\mathcal{E}) & \xrightarrow{\bullet^2} & \overline{A}_\infty^u(F\mathcal{C}, F\mathcal{E}) \end{array}$$

that is, to a usual strict 2-functor  $\overline{F} : \overline{A_\infty^u} \rightarrow \overline{A_\infty^u}$ .

Let us show that an  $A_\infty^u$ -2-transformation  $\lambda : F \rightarrow G : A_\infty^u \rightarrow A_\infty^u$  as in Definition 3.2 induces an ordinary strict 2-transformation  $\overline{\lambda} : \overline{F} \rightarrow \overline{G} : \overline{A_\infty^u} \rightarrow \overline{A_\infty^u}$  in cohomology. Indeed, diagram (3.2.1) implies commutativity of diagram

$$\begin{array}{ccc} sA_\infty^u(\mathcal{C}, \mathcal{D}) & \xrightarrow{F_1} & sA_\infty^u(F\mathcal{C}, F\mathcal{D}) \\ G_1 \downarrow & = & \downarrow (1 \boxtimes \lambda_{\mathcal{D}})M_{10} \\ sA_\infty^u(G\mathcal{C}, G\mathcal{D}) & \xrightarrow{(\lambda_{\mathcal{C}} \boxtimes 1)M} & sA_\infty^u(F\mathcal{C}, G\mathcal{D}) \end{array}$$

$$r : f \rightarrow g : G\mathcal{C} \rightarrow G\mathcal{D} \longmapsto \lambda_{\mathcal{C}}r : \lambda_{\mathcal{C}}f \rightarrow \lambda_{\mathcal{C}}g : F\mathcal{C} \rightarrow G\mathcal{D}.$$

Passing to cohomology we get

$$\begin{array}{ccc} \overline{A_\infty^u}(\mathcal{C}, \mathcal{D}) & \xrightarrow{H^0(sF_1s^{-1})} & \overline{A_\infty^u}(F\mathcal{C}, F\mathcal{D}) \\ H^0(sG_1s^{-1}) \downarrow & = & \downarrow \text{--}\lambda_{\mathcal{D}} = \overline{A_\infty^u}(F\mathcal{C}, \lambda_{\mathcal{D}}) \\ \overline{A_\infty^u}(G\mathcal{C}, G\mathcal{D}) & \xrightarrow[\overline{A_\infty^u}(\lambda_{\mathcal{C}}, G\mathcal{D})]{\lambda_{\mathcal{C}} \cdot \text{--}} & \overline{A_\infty^u}(F\mathcal{C}, G\mathcal{D}) \end{array}$$

Therefore,  $\overline{\lambda}_{\mathcal{C}} \in \text{Ob } \overline{A_\infty^u}(F\mathcal{C}, G\mathcal{C})$  form a strict 2-transformation  $\overline{\lambda} : \overline{F} \rightarrow \overline{G} : \overline{A_\infty^u} \rightarrow \overline{A_\infty^u}$ .

**3.3. Examples of  $A_\infty^u$ -2-functors.** Let  $\mathcal{A}$  be an  $A_N$ -category,  $1 \leq N \leq \infty$ . It determines an  $A_\infty^u$ -2-functor  $F = A_N(\mathcal{A}, -) : A_\infty^u \rightarrow A_\infty^u$ , given by the following data:

1. the map  $F : \text{Ob } A_\infty^u \rightarrow \text{Ob } A_\infty^u$ ,  $\mathcal{C} \mapsto A_N(\mathcal{A}, \mathcal{C})$  (the category  $A_N(\mathcal{A}, \mathcal{C})$  is unital by [Lyu03, Proposition 7.7]);
2. the unital strict  $A_\infty$ -functor  $F = A_N(\mathcal{A}, -) : A_\infty^u(\mathcal{C}, \mathcal{D}) \rightarrow A_\infty^u(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{D}))$  for each pair  $\mathcal{C}, \mathcal{D}$  of unital  $A_\infty$ -categories (cf. [Lyu03, Propositions 6.2, 8.4]).

Clearly,  $\text{id}_{A_N(\mathcal{A}, \mathcal{C})} = (1 \boxtimes \text{id}_{\mathcal{C}})M = A_N(\mathcal{A}, \text{id}_{\mathcal{C}})$ . We want to prove now that the equation

$$\begin{aligned} [TsA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty^u(\mathcal{D}, \mathcal{E}) &\xrightarrow{M} TsA_\infty^u(\mathcal{C}, \mathcal{E}) \xrightarrow{A_N(\mathcal{A}, -)} TsA_\infty^u(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{E}))] \\ &= [TsA_\infty^u(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty^u(\mathcal{D}, \mathcal{E}) \xrightarrow{A_N(\mathcal{A}, -) \boxtimes A_N(\mathcal{A}, -)} \\ &\quad TsA_\infty^u(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{D})) \boxtimes TsA_\infty^u(A_N(\mathcal{A}, \mathcal{D}), A_N(\mathcal{A}, \mathcal{E})) \\ &\quad \xrightarrow{M} TsA_\infty^u(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{E}))] \quad (3.3.1) \end{aligned}$$

holds strictly for each triple  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  of unital  $A_\infty$ -categories. In fact, this  $F$  is a restriction of an  $A_\infty$ -2-functor  $F : \text{Ob } A_\infty \rightarrow \text{Ob } A_\infty$ ,  $\mathcal{C} \mapsto A_N(\mathcal{A}, \mathcal{C})$ , which is defined just as in Definition 3.1 without mentioning the unitality. Equation (3.3.1) follows from a similar

equation without the unitality index  $u$ . To prove it we consider the compositions

$$\begin{aligned}
& [TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \xrightarrow{1 \boxtimes M} TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{E}) \\
& \xrightarrow{1 \boxtimes A_N(\mathcal{A}, -)} TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{E})) \xrightarrow{\alpha} TsA_N(\mathcal{A}, \mathcal{E})] \\
& = [TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \\
& \xrightarrow{1 \boxtimes M} TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{E}) \xrightarrow{M} TsA_N(\mathcal{A}, \mathcal{E})] \\
& = [TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \\
& \xrightarrow{M \boxtimes 1} TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{E}) \xrightarrow{M} TsA_N(\mathcal{A}, \mathcal{E})] \\
& = [TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \xrightarrow{1 \boxtimes A_N(\mathcal{A}, -) \boxtimes 1} \\
& TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{D})) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \\
& \xrightarrow{\alpha \boxtimes 1} TsA_N(\mathcal{A}, \mathcal{D}) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \\
& \xrightarrow{1 \boxtimes A_N(\mathcal{A}, -)} TsA_N(\mathcal{A}, \mathcal{D}) \boxtimes TsA_\infty(A_N(\mathcal{A}, \mathcal{D}), A_N(\mathcal{A}, \mathcal{E})) \xrightarrow{\alpha} TsA_N(\mathcal{A}, \mathcal{E})] \\
& = [TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(\mathcal{C}, \mathcal{D}) \boxtimes TsA_\infty(\mathcal{D}, \mathcal{E}) \xrightarrow{1 \boxtimes A_N(\mathcal{A}, -) \boxtimes A_N(\mathcal{A}, -)} \\
& TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{D})) \boxtimes TsA_\infty(A_N(\mathcal{A}, \mathcal{D}), A_N(\mathcal{A}, \mathcal{E})) \\
& \xrightarrow{1 \boxtimes M} TsA_N(\mathcal{A}, \mathcal{C}) \boxtimes TsA_\infty(A_N(\mathcal{A}, \mathcal{C}), A_N(\mathcal{A}, \mathcal{E})) \xrightarrow{\alpha} TsA_N(\mathcal{A}, \mathcal{E})].
\end{aligned}$$

By Proposition 1.6 we deduce equation (3.3.1) (see also [Lyu03, Proposition 5.5]).

Let now  $\mathcal{A}$  be a unital  $A_\infty$ -category. It determines an  $A_\infty^u$ -2-functor  $G = A_\infty^u(\mathcal{A}, -) : A_\infty^u \rightarrow A_\infty^u$ , given by the following data:

1. the map  $G : \text{Ob } A_\infty^u \rightarrow \text{Ob } A_\infty^u$ ,  $\mathcal{C} \mapsto A_\infty^u(\mathcal{A}, \mathcal{C})$  (the category  $A_\infty^u(\mathcal{A}, \mathcal{C})$  is unital by [Lyu03, Proposition 7.7]);
2. the unital strict  $A_\infty$ -functor  $G = A_\infty^u(\mathcal{A}, -) : A_\infty^u(\mathcal{C}, \mathcal{D}) \rightarrow A_\infty^u(A_\infty^u(\mathcal{A}, \mathcal{C}), A_\infty^u(\mathcal{A}, \mathcal{D}))$  for each pair  $\mathcal{C}, \mathcal{D}$  of unital  $A_\infty$ -categories, determined from

$$\begin{aligned}
M &= [TsA_\infty^u(\mathcal{A}, \mathcal{B}) \boxtimes TsA_\infty^u(\mathcal{B}, \mathcal{C}) \xrightarrow{1 \boxtimes A_\infty^u(\mathcal{A}, -)} \\
& TsA_\infty^u(\mathcal{A}, \mathcal{B}) \boxtimes TsA_\infty^u(A_\infty^u(\mathcal{A}, \mathcal{B}), A_\infty^u(\mathcal{A}, \mathcal{C})) \xrightarrow{\alpha} TsA_\infty^u(\mathcal{A}, \mathcal{C})].
\end{aligned}$$

(cf. [Lyu03, Propositions 6.2, 8.4]).

Clearly,  $G\mathcal{C}$  are full  $A_\infty$ -subcategories of  $F\mathcal{C}$  for the  $A_\infty^u$ -2-functor  $F = A_\infty(\mathcal{A}, -)$ . Furthermore,  $A_\infty$ -functors  $G_{\mathcal{C}, \mathcal{D}}(f)$  are restrictions of  $A_\infty$ -functors  $F_{\mathcal{C}, \mathcal{D}}(f)$ , so  $G$  is a full  $A_\infty^u$ -2-subfunctor of  $F$ . In particular,  $G$  satisfies equation (3.1.1). Another way to prove that  $G$  is an  $A_\infty^u$ -2-functor is to repeat the reasoning concerning  $F$ .

**3.4. Example of an  $A_\infty^u$ -2-equivalence.** Assume that  $\mathcal{Q}$  is a differential graded  $\mathbb{k}$ -quiver. As usual,  $\mathcal{F}\mathcal{Q}$  denotes the free  $A_\infty$ -category generated by it. We claim that  $\text{restr} :$

$A_\infty(\mathcal{FQ}, \_) \rightarrow A_1(\mathcal{Q}, \_) : A_\infty^u \rightarrow A_1^u$  is a strict 2-natural  $A_\infty$ -equivalence. Indeed, it is given by the family of unital  $A_\infty$ -functors  $\text{restr}_\mathcal{C} : A_\infty(\mathcal{FQ}, \mathcal{C}) \rightarrow A_1(\mathcal{Q}, \mathcal{C})$ ,  $\mathcal{C} \in \text{Ob } A_\infty^u$ , which are equivalences by Theorem 2.12. We have to prove that the diagram of  $A_\infty$ -functors

$$\begin{array}{ccc} A_\infty^u(\mathcal{C}, \mathcal{D}) & \xrightarrow{A_\infty(\mathcal{FQ}, \_)} & A_\infty^u(A_\infty(\mathcal{FQ}, \mathcal{C}), A_\infty(\mathcal{FQ}, \mathcal{D})) \\ \downarrow A_1(\mathcal{Q}, \_) & = & \downarrow (1 \boxtimes \text{restr}_\mathcal{D})M \\ A_\infty^u(A_1(\mathcal{Q}, \mathcal{C}), A_1(\mathcal{Q}, \mathcal{D})) & \xrightarrow{(\text{restr}_\mathcal{C} \boxtimes 1)M} & A_\infty^u(A_\infty(\mathcal{FQ}, \mathcal{C}), A_1(\mathcal{Q}, \mathcal{D})) \end{array} \quad (3.4.1)$$

commutes. Notice that all arrows in this diagram are strict  $A_\infty$ -functors. Indeed,  $A_\infty(\mathcal{FQ}, \_)$  and  $A_1(\mathcal{Q}, \_)$  are strict by [Lyu03, Proposition 6.2]. For an arbitrary  $A_\infty$ -functor  $f$  the components  $[(f \boxtimes 1)M]_n = (f \boxtimes 1)M_{0n}$  vanish for all  $n$  except for  $n = 1$ , thus,  $(f \boxtimes 1)M$  is strict. The  $A_\infty$ -functor  $g = \text{restr}_\mathcal{D}$  is strict, hence, the  $n$ -th component

$$[(1 \boxtimes g)M]_n : r^1 \otimes \cdots \otimes r^n \mapsto (r^1 \otimes \cdots \otimes r^n \mid g)M_{n0}$$

of the  $A_\infty$ -functor  $(1 \boxtimes g)M$  satisfies the equation

$$[(r^1 \otimes \cdots \otimes r^n \mid g)M_{n0}]_k = (r^1 \otimes \cdots \otimes r^n)\theta_{k1}g_1.$$

If the right hand side does not vanish, then  $n \leq 1 \leq k + n$ , so  $n = 1$  and  $(1 \boxtimes g)M$  is strict.

Given an  $A_\infty$ -transformation  $t : g \rightarrow h : \mathcal{C} \rightarrow \mathcal{D}$  between unital  $A_\infty$ -functors we find that

$$\begin{aligned} A_\infty(\mathcal{FQ}, \_)(t) &= [(1 \boxtimes t)M : (1 \boxtimes g)M \rightarrow (1 \boxtimes h)M : A_\infty(\mathcal{FQ}, \mathcal{C}) \rightarrow A_\infty(\mathcal{FQ}, \mathcal{D})], \\ A_1(\mathcal{Q}, \_)(t) &= [(1 \boxtimes t)M : (1 \boxtimes g)M \rightarrow (1 \boxtimes h)M : A_1(\mathcal{Q}, \mathcal{C}) \rightarrow A_1(\mathcal{Q}, \mathcal{D})], \\ [(1 \boxtimes \text{restr}_\mathcal{D})M]A_\infty(\mathcal{FQ}, \_)(t) &= [((1 \boxtimes t)M) \cdot \text{restr}_\mathcal{D} : ((1 \boxtimes g)M) \cdot \text{restr}_\mathcal{D} \\ &\quad \rightarrow ((1 \boxtimes h)M) \cdot \text{restr}_\mathcal{D} : A_\infty(\mathcal{FQ}, \mathcal{C}) \rightarrow A_1(\mathcal{Q}, \mathcal{D})], \\ [(\text{restr}_\mathcal{C} \boxtimes 1)M]A_1(\mathcal{Q}, \_)(t) &= [\text{restr}_\mathcal{C} \cdot ((1 \boxtimes t)M) : \text{restr}_\mathcal{C} \cdot ((1 \boxtimes g)M) \\ &\quad \rightarrow \text{restr}_\mathcal{C} \cdot ((1 \boxtimes h)M) : A_\infty(\mathcal{FQ}, \mathcal{C}) \rightarrow A_1(\mathcal{Q}, \mathcal{D})]. \end{aligned}$$

We have to verify that the last two  $A_\infty$ -transformations are equal. First of all, let us show that mappings of objects in (3.4.1) commute. Given a unital  $A_\infty$ -functor  $g : \mathcal{C} \rightarrow \mathcal{D}$ , we are going to check that

$$[(1 \boxtimes g)M]_n \cdot \text{restr}_1 = \text{restr}_1^{\otimes n} \cdot [(1 \boxtimes g)M]_n \quad (3.4.2)$$

for any  $n \geq 1$ . Indeed, for any  $n$ -tuple of composable  $A_\infty$ -transformations

$$f^0 \xrightarrow{r^1} f^1 \longrightarrow \cdots \xrightarrow{r^n} f^n : \mathcal{FQ} \rightarrow \mathcal{C},$$

we have in both cases

$$\begin{aligned}
\{(r^1 \otimes \cdots \otimes r^n)[(1 \boxtimes g)M]_n\}_0 &= [(r^1 \otimes \cdots \otimes r^n|g)M_{n0}]_0 = (r_0^1 \otimes \cdots \otimes r_0^n)g_n, \\
\{(r^1 \otimes \cdots \otimes r^n)[(1 \boxtimes g)M]_n\}_1 &= [(r^1 \otimes \cdots \otimes r^n|g)M_{n0}]_1 \\
&= \sum_{i=1}^n (r_0^1 \otimes \cdots \otimes r_0^{i-1} \otimes r_1^i \otimes r_0^{i+1} \otimes \cdots \otimes r_0^n)g_n \\
&\quad + \sum_{i=1}^n (r_0^1 \otimes \cdots \otimes r_0^{i-1} \otimes f_1^i \otimes r_0^{i+1} \otimes \cdots \otimes r_0^n)g_{n+1}.
\end{aligned}$$

Note that the right hand sides depend only on 0-th and 1-st components of  $r^i$ ,  $f^i$ . This is precisely what is claimed by equation (3.4.2).

The coincidence of  $A_\infty$ -transformations  $((1 \boxtimes t)M) \cdot \text{restr}_{\mathcal{D}} = \text{restr}_{\mathcal{C}} \cdot ((1 \boxtimes t)M)$  follows similarly from the computation:

$$\begin{aligned}
\{(r^1 \otimes \cdots \otimes r^n)[(1 \boxtimes t)M]_n\}_0 &= [(r^1 \otimes \cdots \otimes r^n \boxtimes t)M_{n1}]_0 = (r_0^1 \otimes \cdots \otimes r_0^n)t_n, \\
\{(r^1 \otimes \cdots \otimes r^n)[(1 \boxtimes t)M]_n\}_1 &= [(r^1 \otimes \cdots \otimes r^n \boxtimes t)M_{n1}]_1 \\
&= \sum_{i=1}^n (r_0^1 \otimes \cdots \otimes r_0^{i-1} \otimes r_1^i \otimes r_0^{i+1} \otimes \cdots \otimes r_0^n)t_n \\
&\quad + \sum_{i=1}^n (r_0^1 \otimes \cdots \otimes r_0^{i-1} \otimes f_1^i \otimes r_0^{i+1} \otimes \cdots \otimes r_0^n)t_{n+1}.
\end{aligned}$$

**3.5. Representability.** An  $A_\infty^u$ -2-functor  $F : A_\infty^u \rightarrow A_\infty^u$  is called *representable*, if it is naturally  $A_\infty^u$ -2-equivalent to the  $A_\infty^u$ -2-functor  $A_\infty(\mathcal{A}, -) : A_\infty^u \rightarrow A_\infty^u$  for some  $A_\infty$ -category  $\mathcal{A}$ . The above results imply that the  $A_\infty^u$ -2-functor  $A_1(\mathcal{Q}, -)$  corresponding to a differential graded  $\mathbb{k}$ -quiver  $\mathcal{Q}$  is represented by the free  $A_\infty$ -category  $\mathcal{FQ}$  generated by  $\mathcal{Q}$ .

This definition of representability has a disadvantage: many different  $A_\infty$ -categories can represent the same  $A_\infty^u$ -2-functor. More attractive notion is the following. An  $A_\infty^u$ -2-functor  $F : A_\infty^u \rightarrow A_\infty^u$  is called *unitally representable*, if it is naturally  $A_\infty^u$ -2-equivalent to the  $A_\infty^u$ -2-functor  $A_\infty^u(\mathcal{A}, -) : A_\infty^u \rightarrow A_\infty^u$  for some unital  $A_\infty$ -category  $\mathcal{A}$ . Such  $\mathcal{A}$  is unique up to an  $A_\infty$ -equivalence. Indeed, composing a natural 2-equivalence  $\bar{\lambda} : A_\infty^u(\mathcal{A}, -) \rightarrow A_\infty^u(\mathcal{B}, -) : \overline{A_\infty^u} \rightarrow \overline{A_\infty^u}$  with the 0-th cohomology 2-functor  $H^0 : \overline{A_\infty^u} \rightarrow \mathcal{Cat}$ , we get a natural 2-equivalence  $H^0 \bar{\lambda} : H^0 \overline{A_\infty^u}(\mathcal{A}, -) \rightarrow H^0 \overline{A_\infty^u}(\mathcal{B}, -) : \overline{A_\infty^u} \rightarrow \mathcal{Cat}$ . However,  $H^0 \overline{A_\infty^u}(\mathcal{A}, -) = \overline{A_\infty^u}(\mathcal{A}, -)$ , so using a 2-category version of Yoneda lemma one can deduce that  $\mathcal{A}$  and  $\mathcal{B}$  are equivalent in  $\overline{A_\infty^u}$ . We shall present an example of unital representability in subsequent publication [LM04].

**3.6. Acknowledgements.** We are grateful to all the participants of the  $A_\infty$ -category seminar at the Institute of Mathematics, Kyiv, for attention and fruitful discussions, especially to Yu. Bєspalov and S. Ovsienko. One of us (V.L.) is grateful to Max-Planck-Institut für Mathematik for warm hospitality and support at the final stage of this research.

## References

- [Fuk93] K. Fukaya, *Morse homotopy,  $A_\infty$ -category, and Floer homologies*, Proc. of GARC Workshop on Geometry and Topology '93 (H. J. Kim, ed.), Lecture Notes, no. 18, Seoul Nat. Univ., Seoul, 1993, pp. 1–102.
- [Fuk02] K. Fukaya, *Floer homology and mirror symmetry. II*, Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999), Adv. Stud. Pure Math., vol. 34, Math. Soc. Japan, Tokyo, 2002, pp. 31–127.
- [Kad82] T. V. Kadeishvili, *The algebraic structure in the homology of an  $A(\infty)$ -algebra*, Soobshch. Akad. Nauk Gruzin. SSR **108** (1982), no. 2, 249–252, in Russian.
- [Kel01] B. Keller, *Introduction to  $A$ -infinity algebras and modules*, Homology, Homotopy and Applications **3** (2001), no. 1, 1–35, [arXiv:math.RA/9910179](http://arxiv.org/abs/math.RA/9910179), <http://intlpress.com/HHA/v3/n1/a1/>.
- [Kon95] M. Kontsevich, *Homological algebra of mirror symmetry*, Proc. Internat. Cong. Math., Zürich, Switzerland 1994 (Basel), vol. 1, Birkhäuser Verlag, 1995, 120–139.
- [KS02] M. Kontsevich and Y. S. Soibelman,  *$A_\infty$ -categories and non-commutative geometry*, in preparation, 2002.
- [KS] M. Kontsevich and Y. S. Soibelman, *Deformation theory*, book in preparation.
- [LH03] K. Lefèvre-Hasegawa, *Sur les  $A_\infty$ -catégories*, Ph.D. thesis, Université Paris 7, U.F.R. de Mathématiques, 2003, [arXiv:math.CT/0310337](http://arxiv.org/abs/math.CT/0310337).
- [LM04] V. V. Lyubashenko and O. Manzyuk, *Quotients of unital  $A_\infty$ -categories*, 2004, [arXiv:math.CT/0306018](http://arxiv.org/abs/math.CT/0306018).
- [LO02] V. V. Lyubashenko and S. A. Ovsienko, *A construction of quotient  $A_\infty$ -categories*, Homology, Homotopy Appl. **8** (2006), no. 2, 157–203, [arXiv:math.CT/0211037](http://arxiv.org/abs/math.CT/0211037) <http://intlpress.com/HHA/v8/n2/a9/>.
- [Lyu03] V. V. Lyubashenko, *Category of  $A_\infty$ -categories*, Homology, Homotopy and Applications **5** (2003), no. 1, 1–48, [arXiv:math.CT/0210047](http://arxiv.org/abs/math.CT/0210047), <http://intlpress.com/HHA/v5/n1/a1/>.
- [Sta63] J. D. Stasheff, *Homotopy associativity of  $H$ -spaces, I & II*, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312.

# Contents

<b>1</b>	<b>Conventions and preliminaries</b>	<b>2</b>
1.1	$A_N$ -categories. . . . .	3
1.7	Trees. . . . .	8
<b>2</b>	<b>Properties of free <math>A_\infty</math>-categories</b>	<b>9</b>
2.1	Construction of a free $A_\infty$ -category. . . . .	9
2.6	Explicit formula for the constructed strict $A_\infty$ -functor. . . . .	15
2.7	Transformations between functors from a free $A_\infty$ -category. . . . .	15
2.11	Restriction as an $A_\infty$ -functor. . . . .	22
<b>3</b>	<b>Representable 2-functors <math>A_\infty^u \rightarrow A_\infty^u</math></b>	<b>24</b>
3.3	Examples of $A_\infty^u$ -2-functors. . . . .	27
3.4	Example of an $A_\infty^u$ -2-equivalence. . . . .	28
3.5	Representability. . . . .	30
3.6	Acknowledgements. . . . .	30